Emory University

MATH 250 - Foundations of Mathematics Learning Notes

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Contents

1 Mathematical Reasoning

1.1 Statement

Definition 1.1.1 (Statement). A **statement** is any declarative sentence that is either true or false.

Remark. A statement cannot be ambiguous, cannot be both true and false, and cannot be sometimes true or false.

Example 1.1.1. Examples and Non-Examples:

- All integers are rational numbers. – True statement
- π is irrational. – True statement
- $1 = 0$. – False statement
- $\sqrt{2} \in \mathbb{Z}$. – False statement
- Every student in this class is a math major. – False statement. (To prove this false, find a student that is not a math major.)
- Solve the equation $2x = 3$. – Not a statement
- Chocolate chip is the best ice cream flavor. – Not a statement
- $x + 5 = 3$. – Not a statement. (Turn it into a statement: When $x = 1$, $x + 5$=3; or when $x \neq -2,\ x + 5 \neq 3$.)

Remark. $\in$ means belongs to, and $\mathbb{Z}$ is the notation for integers.

Remark. We denote statements by letters.

Example 1.1.2. P: "Today is a sunny day;" Q: "3 is a prime number.

If the statement's truth depends on a variable, we include the variable in the notation.

Example 1.1.3. $R(x)$: "x is an integer;" $P(x)$: $x + 5 = 3$. $P(x)$ where $x \neq -2$ is false; $P(-2)$ is true.

Example 1.1.4. $R(f)$: "f is an increasing function."

- If $f(x) = e^x$, then $R(f)$ is true.
- If $f(x) = x^2$, then $R(f)$ is false.

Notation 1.1 (Quantifiers). We use the symbol $\forall$ for "for all" or "for any" and the symbol $\exists$ for "there exists." $\forall$ and $\exists$ are called **quantifiers**. $\forall$ is the **universal quantifier** and $\exists$ is the **existential quantifier**.

Example 1.1.5.

1. For any $\varepsilon > 0$, there is a $\delta > 0$.
$$\forall \varepsilon > 0,\ \exists \delta > 0.$$
2. The square of every real number is non-negative.
$$\forall x \in \mathbb{R},\ x^2 \geq 0.$$
3. There is a real number whose square is negative.
$$\exists x \in \mathbb{R} \textit{ s.t. } x^2 < 0.$$
4. For every real number x, there is an integer n such that $n > x$.
$$\forall x \in \mathbb{R},\ \exists n \in \mathbb{Z} \textit{ s.t. } n > x.$$
5. For all rational numbers x, there exist integers a and b such that $x = \frac{a}{b}$.
$$\forall x \in \mathbb{Q},\ \exists a, b \in \mathbb{Z} \textit{ s.t. } x = \frac{a}{b}.$$

Example 1.1.6.

1. $\forall n \in \mathbb{Z},\ \exists m \in \mathbb{Z},\ \textit{s.t. } m = n+5$: For every integers n, there exists integers m such that $m = n + 5$.
2. $\forall \varepsilon > 0,\ \exists \delta > 0,\ \textit{s.t. } \forall x, y \in \mathbb{R},$ we have $|x - y| < \varepsilon \implies |x^2 - y^2| < \delta$: For every ε greater than 0, there exists a δ greater than 0 such that for all real x and y with $|x - y|$ les than ε, we have $|x^2 - y^2|$ is less than δ.

Remark. $\forall \varepsilon \exists \delta$ and $\exists \delta \forall \varepsilon$ are not the same.
In $\forall \varepsilon \exists \delta$, δ depends on ε, but δ is independent in $\exists \delta \forall \varepsilon$, meaning the same δ works for all ε.

Example 1.1.7. Compare $\forall n \in \mathbb{Z},\ \exists m \in \mathbb{Z} \textit{ s.t. } m = n^2$ to $\exists n \in \mathbb{Z} \textit{ s.t. } \forall m \in \mathbb{Z},\ m = n^2$.

Answer.

1. $\forall n \in \mathbb{Z},\ \exists m \in \mathbb{Z} \textit{ s.t. } m = n^2$: True. The square of an integer is an integer.
2. $\exists n \in \mathbb{Z} \textit{ s.t. } \forall m \in \mathbb{Z},\ m = n^2$.: False: n^2 is just one integer and cannot be represented by all $m \in \mathbb{Z}$.

$\square$

Definition 1.1.2 (Negations). If P is a statement, the **negation** of P, written as $\neg P$ (read as "not P" or "negation of P") is the statement "P is false."

$$\begin{cases} P \text{ is true} \\ \neg P \text{ is false} \end{cases} \qquad \begin{cases} P \text{ is false} \\ \neg P \text{ is true} \end{cases}$$

Example 1.1.8. Write the negation of P: "All apples are fruits" and check the truth value of both P and $\neg P$.

Answer. $\neg P$: Not all apples are fruits / Some apples are not fruits / There exists an apple that is not a fruit.

P is true and $\neg P$ is false. □

Example 1.1.9. Write the negation of P: "Everyday this week was hot."

Answer. $\neg P$: Somedays this week were not hot. / Not all days this week were hot. / There was a day this week that was not hot. □

Example 1.1.10. P all primes are odd.

$\neg P$: There exists ($\exists$) a prime that is even / Some primes are not odd. (True: 2 is even.)

Example 1.1.11. Q: $\forall x \in \mathbb{R},\ x^2 > x$

$\neg Q$: $\exists x \in \mathbb{R},$ ***s.t.*** $x^2 \leq x$ (True: for $x = \dfrac{1}{2},\ x^2 = \frac{1}{4} < x$)

Example 1.1.12. R: There exists a real solution to $x^2 + 1 = 0$ / $\exists x \in \mathbb{R}$ ***s.t.*** $x^2 + 1 = 0$

$\neg R$: $\forall x \in \mathbb{R},\ x^2 + 1 \neq 0$ – (True .)

Remark (Negating Quantifiers). In general:

$$\neg(\forall x \in S, P(x)) = \exists x \in S \textbf{\textit{ s.t. }} \neg P(x).$$

$$\neg(\exists x \in S, P(x)) = \forall x \in S, \neg P(x).$$

Example 1.1.13. $\exists n > 0$ ***s.t.*** $\forall m > 0, m < n$

Negation: $\forall n > 0, \exists m > 0$ ***s.t.*** $m \geq n.$

Example 1.1.14. For all $x \in \mathbb{R}$, there exists a $y \in \mathbb{R}$, ***s.t.*** $xy = 1$

Negation: $\exists x \in \mathbb{R}$ ***s.t.*** $\forall y \in \mathbb{R},\ xy \neq 1$

1.2 Compound Statements

Definition 1.2.1 (Conjunction and Disjunction). The **conjunction** of P and Q, denoted $P \wedge Q$, is the statement "Both P and Q are true." The **disjunction** of P and Q, denoted $P \vee Q$, is the statement "P is true or Q is true."

Example 1.2.1. Which of the following are true statements?

1. The function x^2 is even and it is concave up.

   P: x^2 is even (True), Q: x^2 is concave up (True).

   $$\therefore P \wedge Q \text{ is true.}$$

2. Every student in this class is a math major and a human being.

   P: Every student in this class is a math major (False), Q: Every student in this class is a human being (True).

   $$\therefore P \wedge Q \text{ is false.}$$

3. Every student in this class is a math major or a human being.

   $$P \vee Q \text{ is true.}$$

Example 1.2.2. The following inequality can be written in conjunctions or disjunctions.

1. $|x| < 3$: $(-3 < x < 3)$ Conjunction: $x > -3$ AND $x < 3$

2. $|x| > 3$: Disjunction: $x < -3$ OR $x > 3$

Remark. $\vee$ is **inclusive or**, meaning if both P and Q are true, then $P \vee Q$ is also true, as opposed to **exclusive or**, meaning "either P or Q but not both."

Definition 1.2.2 (Truth Tables). **Truth tables** are a handy way to organize information.

Example 1.2.3. Write the truth table for the statement form: $P \vee \neg Q$.

Answer.

P	Q	$\neg Q$	$P \vee \neg Q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

□

Definition 1.2.3 (Logically Equivalent). Suppose P and Q are statements. We say P and Q are **logically equivalent**, denoted as $P \equiv Q$, if they are either both true or both false.

Note: If two statements are logically equivalent, their truth values match up line by line in a truth table. We use this techniques to prove two statements mathematically (logically) equivalent.

Example 1.2.4. Proving the following using truth tables.

1. $\neg(\neg P) \equiv P$.

   Answer.

P	$\neg P$	$\neg(\neg P)$
T	F	T
F	T	F

   □

2. $P \vee Q \equiv Q \vee P$.

   Answer.

P	Q	$P \vee Q$	$Q \vee P$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

   □

3. $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$.

   Answer.

P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$\neg(P \vee Q)$	$(\neg P) \wedge (\neg Q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

   □

4. $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$.

   Answer.

P	Q	R	$Q \vee R$	$P \vee (Q \vee R)$	$P \vee Q$	$(P \vee Q) \vee R$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T
F	F	F	F	F	F	F

□

Theorem 1.2.1 (Negating Compound Statements).

$$\neg(P \wedge Q) = \neg P \vee \neg Q$$

$$\neg(P \vee Q) = \neg P \wedge \neg Q$$

Example 1.2.5. It is not true that the numbers x and y are both even.

P: x is even; Q: y is even.

$\neg(P \wedge Q)$: x is odd ($\neg P$) OR y is odd ($\neg Q$).

Example 1.2.6. It is not true that 9 is prime or 9 is even.

P: 9 is prime; Q: 9 is even.

$\neg(P \vee Q)$: 9 is not prime ($\neg P$) AND 9 is odd ($\neg Q$)

Theorem 1.2.2 (Distributivity).

$$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$$

Remark. Truth tables can balloon in size. Indeed, if you have $P_1, \cdots, P_n$ statements involved in your statement form, your truth table will contain 2^n rows.

Sometimes, prove logical equivalences without using truth tables by taking advantage of the logical equivalences already proven is more approachable.

Example 1.2.7. Prove or disprove without using ruth tables.

1. $\neg(P \wedge (\neg Q)) \equiv (\neg P) \vee Q$.

   ***Proof* (1).**

$$\begin{aligned} \neg(P \wedge (\neg Q)) &\equiv \neg P \vee (\neg(\neg Q)) \\ &\equiv \neg P \vee Q \qquad \text{[Negation of } \neg Q \equiv Q\text{]} \end{aligned}$$

■

2. $P \wedge ((Q \vee R) \vee S) \equiv (P \wedge Q) \vee (P \wedge R) \vee (P \wedge S)$.

   ***Proof* (2).**

$$\begin{aligned} P \wedge ((Q \vee R) \vee S) &\equiv (P \wedge (Q \vee R)) \vee (P \wedge S) && \text{[Distributivity]} \\ &\equiv (P \wedge Q) \vee (P \wedge R) \vee (P \wedge S) && \text{[Distributivity]} \end{aligned}$$

■

3. $P \vee ((Q \wedge R) \wedge S) \equiv (P \vee Q) \wedge (P \vee R) \wedge (P \vee S)$.

   ***Proof* (3).**

$$\begin{aligned} P \vee ((Q \wedge R) \wedge S) &\equiv (P \vee (Q \wedge R)) \wedge (P \vee S) && \text{[Distributivity]} \\ &\equiv (P \vee Q) \wedge (P \vee R) \wedge (P \vee S) && \text{[Distributivity]} \end{aligned}$$

■

Definition 1.2.4 (Tautology). A statement form that is true in all possible cases (for example, regardless of the truth values of P and Q) is called a **tautology**.

Definition 1.2.5 (Contradiction). A statement that is false in all possible cases (for example, regardless of the truth values for P and Q) is called a **contradiction**.

Example 1.2.8. Classify each of the following as a tautology, a contradiction, or neither.

1. $P \vee \neg P$

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

$\therefore$ Tautology.

2. $P \wedge \neg P$

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

$\therefore$ Contradiction.

3. $P \vee Q$

P	A	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

$\therefore$ Neither.

1.3 Implications

Definition 1.3.1 (Implications or Conditional Statements). The symbol $\Rightarrow$ means "implies." So $P \Rightarrow Q$ means "P implies Q" or "If P, then Q."

Example 1.3.1.

- If this is an apple, then it is a fruit.
- If a function f is differentiable, then it is continuous.

Remark. In mathematics, the truth value of an implication is **not** determined by causality. That is, the statement $P \Rightarrow Q$ means that in all circumstances in which P is true, Q is also true.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

The only way for $P \Rightarrow Q$ to be false is when P is true but Q is false. This implies that false assumptions can lead to any conclusions.

Definition 1.3.2. Sufficient Condition and Necessary Condition If $P \Rightarrow Q$ is true, then P is called a **sufficient condition** for Q, and Q is called a **necessary condition** for P. There are many different ways to write $P \Rightarrow Q$in English:

$$P \implies Q \begin{cases} \text{If } P, \text{ then } Q. \\ Q \text{ if } P. \\ Q \text{ whenever } P. \\ Q, \text{ provided that } P \\ \text{Whenever } P, \text{ then also } Q. \\ P \text{ is a sufficient condition for } Q. \\ \text{For } Q\text{, it is sufficient that } P. \\ Q \text{ is a necessary condition for } P. \\ \text{For } P\text{, it is necessary that } Q. \\ P \text{ only if } Q. \end{cases}$$

Theorem 1.3.1 (Negation of Implications). Negation of an implication is a conjunction:

$$\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$$

Example 1.3.2. Negate the following:

$$\forall \varepsilon\, \exists \delta > 0,\ |x-y| < \varepsilon \implies |x^2 - y^2| < \delta.$$

Answer.

$$\exists \varepsilon \textit{ s.t. } \forall \delta > 0,\ |x-y| < \varepsilon \text{ and } |x^2 - y^2| \geq \delta.$$

□

Example 1.3.3. Write in symbols and negate the statement P: "For any positive ε there exists a positive M such that $|f(x) - b| < \varepsilon$ whenever $x > M$."

Answer.

$$P:\ \forall \varepsilon > 0,\ \exists M > 0 \textit{ s.t. } x > M \implies |f(x) - b| < \varepsilon.$$

$$\neg P:\ \exists \varepsilon > 0 \textit{ s.t. } \forall M > 0,\ x > M \text{ and } |f(x) - b| \geq \varepsilon.$$

□

Definition 1.3.3 (Axiom). An **axiom** is a statement which is regarded as being established, accepted, or self-evidently true.

Definition 1.3.4 (Theorem). A **theorem** is a statement that is true and has been verified as true.

Definition 1.3.5 (Proposition). A **proposition** is a smaller, less important theorem.

Definition 1.3.6 (Lemma). A **lemma** is a theorem whose main purpose is to help prove another theorem.

Definition 1.3.7 (Corollary). A **corollary** is a result that is the immediate consequence of a theorem.

Remark (Proving a theorem). Want to show (WTS) $P \Rightarrow Q$: If P is false, $P \Rightarrow Q$ is automatically true, so there's nothing to show here. However, if P is true, we need to show Q is true (for $P \Rightarrow Q$ to be true)l.

First line of a direct proof: Suppose P.

Last line of a direct proof: Therefore Q.

Definition 1.3.8 (Divisibility). Let a and b be integers, with a non-zero. We say a **divides** b, written $a \mid b$, if there exists an *integer* k such that $b = ak$.

In this case, we way that a is a **factor** of b, and that b is a **multiple** of a.

Also note that $a \nmid b$ means that a does not divide b. That is, $\nexists k \in \mathbb{Z}$, such that $b = ak$.

Definition 1.3.9 (Even). Let n be integer. We say n is **eve** if $2 \mid n$.

Definition 1.3.10 (Odd). Let n be an integer. We say n is **odd** if $2 \nmid n$, $n = 2k+1$, or $n = 2k-1$ for some (*f.s.*) $k \in \mathbb{Z}$.

Extension. The possible remainders if we divide an integer by 5 is $0,\ 1,\ 2,\ 3,\ 4$, so we can suppose n to be $5k,\ 5k+1,\ 5k+2,\ 5k+3,\ 5k+4$, respectively.

Axiom 1.1. Integers are closed under addition.

Axiom 1.2. If $k \in \mathbb{Z}$, $(-k) \in \mathbb{Z}$.

`Let` a, b `and` c `be integers, with` a `non-zero. If` $a \mid b$ `and` $a \mid c$, `then` $a \mid (b+c)$.

***Proof* (1).**

Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$.

Suppose $a \mid b$ and $a \mid c$. Then, by definition of divides, $\exists k, l \in \mathbb{Z}$ ***s.t.***

$$b = ak \qquad \text{and} \qquad c = al.$$

Then, $b + c = ak + al = a(k+l)$.

Since $k + l \in \mathbb{Z}$ (Axiom 1.1), by definition, $a \mid (b+c)$.

■

`Let` a `and` b `be integers, with` a `non-zero. If` $a \mid b$, `then` $a \mid (-b)$ `and` $(-a) \mid b$.

***Proof* (2).**

Let $a, b \in \mathbb{Z}$, with $a \neq 0$.

Suppose $a \mid b$. Then by definition of divides, $\exists k \in \mathbb{Z}$ ***s.t.*** $b = ak$.

Multiple both sides of this equation by (-1):

$$-b = -ak = a(-k).$$

Since $(-k) \in \mathbb{Z}$, we get $a \mid (-b)$. [Axiom1.2]

Multiple $-b = a(-k)$ by (-1) on both sides:

$$b = (-a)(-k)$$

Since $(-k) \in \mathbb{Z}$, we see that $(-a) \mid b$.

■

`Let` a, b, `and` c `be integers, with` a `and` b `non-zero. If` $(ab) \mid (ac)$, `then` $b \mid c$.

Proof **(3).**

Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$ and $b \neq 0$.

Suppose $(ab) \mid (ac)$. Then $\exists k \in \mathbb{Z}$ ***s.t.*** $ac = (ab)k$.

Divide both sides of the equation by a:

$$c = bk.$$

Since $k \in \mathbb{Z}$, by definition of divides, $b \mid c$.

■

`Let` n `be an integer. If` n `is of the form` $3k+1$ `for some integer` k, `then` n^2 `is again of the form` $4k' + 1$ `for some integer` k'.

Proof **(4).**

Let $n \in \mathbb{Z}$.

Suppose $n = 3k + 1$ *f.s.* $k \in \mathbb{Z}$.

Squaring both sides:

$$\begin{aligned} n^2 = (3k+1)^2 &= 9k^2 + 6k + 1 \\ &= 3(3k^2 + 2k) + 1. \end{aligned}$$

Set $k' = 3k^2 + 2k$.

As $k \in \mathbb{Z}$, $3k^2, 2k \in \mathbb{Z}$, $k' = 3k^2 + 2k \in \mathbb{Z}$.

Then, n^2 is in the form of $3k' + 1$ *f.s.* $k' \in \mathbb{Z}$.

■

`Let` n `and` m `be integers. If` n `and` m `are even, then` $n + m$ `is even.`

Proof **(5).**

Let $n, m \in \mathbb{Z}$.

Suppose n and m are even. Then $\exists s, t \in \mathbb{Z}$ ***s.t.***

$$n = 2s \qquad \text{and} \qquad m = 2t.$$

Then

$$\begin{aligned} n + m &= 2s + 2t \\ &= 2(s + t) \end{aligned}$$

Since $(s + t) \in \mathbb{Z}$, by definition, $n + m$ is even.

■

`Let` n `and` m `be integers. If` m `is even, then` mn `is even.`

***Proof* (6).**
Let $n, m \in \mathbb{Z}$.
Suppose m is even. Then $\exists k \in \mathbb{Z},$ ***s.t.*** $m = 2k$.
So, $mn = (2k)n = 2(kn)$.
Since $(kn) \in \mathbb{Z}$, by definition, mn is even.

■

Definition 1.3.11 (Parity). If x and y have the same **parity**, then x and y both are odd or both are even.

`If` $x \in \mathbb{Z}$, `then` x `and` x^2 `have the same parity.`

Remark. Sometimes, it is helpful to do a case analysis!

***Proof* (7).**
Let $x \in \mathbb{Z}$
$\boxed{\text{Case } 1}$ Suppose x is even.
Then, by definition, $x = 2k$ *f.s.* $k \in \mathbb{Z}$.
So, $x^2 = (2k)^2 = 4k^2 = 2(2k^2)$.
Since $(2k^2) \in \mathbb{Z}$, by definition, x^2 is even.
$\boxed{\text{Case } 2}$ Suppose x is odd.
Then, by definition, $x = 2l + 1$ *f.s.* $l \in \mathbb{Z}$.
So, $x^2 = (2l + 1)^2 = 4l^2 + 4l + 1 = 2(2l^2 + 2l) + 1$
Since $(2l^2 + 2l) \in \mathbb{Z}$, by definition, x^2 is odd.

■

1.4 Contrapositive and Converse

`Let` n `be an integer. Prove that if` $3 \nmid n$, `then` $3 \mid n^2 - 1$.

***Proof* (1).**
Let $n \in \mathbb{Z}$. Suppose $3 \nmid n$.
$\boxed{\text{Case } 1}$ Then $n = 3k + 1$ *f.s.* $k \in \mathbb{Z}$.

$$\begin{aligned} n^2 - 1 &= (3k + 1)^2 - 1 \\ &= 9k^2 + 6k + 1 - 1 \\ &= 9k^2 + 6k \\ &= 3(3k^2 + 2k) \end{aligned}$$

Since $(3k^2+2k) \in \mathbb{Z}$, by definition, $3 \mid n^2-1$.

$\boxed{\text{Case } 2}$ Then $n = 3l+2$ *f.s.* $l \in \mathbb{Z}$.

$$\begin{aligned} n^2-1 &= (3l+2)^2-1 \\ &= 9l^2+18l+4-1 \\ &= 9l^2+18l+3 \\ &= 3(3l^2+6l+1) \end{aligned}$$

Since $(3l^2+6l+1) \in \mathbb{Z}$, by definition, $3 \mid n^2-1$.

■

Sometimes, we can form new implications from old ones. Let P and Q to be statement, and consider the implication $P \Rightarrow Q$.

Definition 1.4.1 (Converse). The **converse** of $P \Rightarrow Q$ is the implication $Q \Rightarrow P$.

Definition 1.4.2 (Contrapositive). The **contrapositive** of $P \Rightarrow Q$ is the implication $\neg Q \Rightarrow \neg P$.

Remark. A question that we are interested in is that "are these equivalent to the original implication?"

Theorem 1.4.1.

$$\neg Q \Rightarrow \neg P \equiv P \Rightarrow Q.$$

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg Q$	$\neg Q \Rightarrow \neg P$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

Since the contrapositive of an implication is logically equivalent to the original implication, we sometimes prove the contrapositive in order to prove the original statement.

`Prove that if the product of two integers` x `and` y `is even, at least one of them must be even.`

***Proof* (2).**
We will prove the contrapositive: "If x and y are both odd, then xy is odd."
Let $x, y \in \mathbb{Z}$. Suppose x and y are odd.
Then $\exists k, l \in \mathbb{Z}$ *s.t.* $x = 2k+1$ and $y = 2l+1$.

So,

$$\begin{aligned} xy = (2k+1)(2l+1) &= 4kl + 2k + 2l + 1 \\ &= 2(2kl + k + l) + 1 \end{aligned}$$

Since $2kl + k + l \in \mathbb{Z}$, we see that xy is odd.

■

`Let` $x \in \mathbb{Z}$. `Prove that if` $x^2 - 4x + 7$ `is even, then` x `is odd.`

Contrapositive: If x is even, then $x^2 - 4x + 7$ is odd.

***Proof* (3).**

We will prove the contrapositive.

Let $x \in \mathbb{Z}$. Suppose x is even. Then $x = 2k$ *f.s.* $k \in \mathbb{Z}$.

Then

$$\begin{aligned} x^2 - 4x + 7 &= (2k)^2 - 4(2k) + 7 \\ &= 4k^2 - 8k + 7 \\ &= 2(2k^2 - 4k + 3) + 1 \end{aligned}$$

Since $2k^2 - 4k + 3 \in \mathbb{Z}$, by definition, $x^2 - 4x + 7$ is odd.

■

`Let` $n \in \mathbb{Z}$. `Prove that if` n^2 `is even, then` n `is even.`

Contrapositive: If n is odd, then n^2 is odd.

Remark. This is a special case of Proof (2), where $x = y$ and is odd.

***Proof* (4).**

We will prove the contrapositive.

Let $n \in \mathbb{Z}$. Suppose n is odd. Then $n = 2k + 1$ *f.s.* $k \in \mathbb{Z}$.

Then

$$\begin{aligned} n^2 = (2k+1)^2 &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

Since $(2k^2 + 2k) \in \mathbb{Z}$, we see n^2 is odd.

■

`Prove that every even number is of the form` $4k$ `or` $4k+2$ `for some` $k \in \mathbb{Z}$.

***Proof* (5).**

We will prove the contrapositive: If x is of the form $4k + 1$ or $4k + 3$, then x is odd.

Suppose $x \in \mathbb{Z}$.

$\boxed{\text{Case } 1}$ Suppose $x = 4k + 1$ *f.s.* $k \in \mathbb{Z}$.

Then $x = 4k + 1 = 2(2k) + 1$.

Since $2k \in \mathbb{Z}$, by definition, x is odd.

$\boxed{\text{Case } 2}$ Suppose $x = 4k + 3$ *f.s.* $k \in \mathbb{Z}$.

Then $x = 4k + 3 = 2(2k + 1) + 1$

Since $2k + 1 \in \mathbb{Z}$, by definition, x is odd.

■

`Prove that every odd number is of the form` $4k+1$ `or` $4k+3$ `for some` $k \in \mathbb{Z}$.

Remark. This implication is the converse of as the contrapositive of the previous one. Here, the implication and the converse of it is both true.

***Proof* (6).**

We will prove the contrapositive: If x is of the form $4k$ or $4k + 2$, then x is even.

Suppose $x \in \mathbb{Z}$.

$\boxed{\text{Case } 1}$ Suppose $x = 4k$ *f.s.* $k \in \mathbb{Z}$.

Then $x = 4k = 2(2k)$.

Since $2k \in \mathbb{Z}$, by definition, x is even.

$\boxed{\text{Case } 2}$ Suppose $x = 4k + 2$ *f.s.* $k \in \mathbb{Z}$.

Then $x = 4k+ = 2 = 2(2k + 1)$

Since $2k + 1 \in \mathbb{Z}$, by definition, x is even.

■

Theorem 1.4.2 (Prove by Contradiction). To prove P, we assume $\neg P$ and show that this implies an absurd statement (like $1 = 0$ or $0 < 0$).

Definition 1.4.3 (Simplest-form). Let m and n be integers, with n non-zero. We say the fraction $\dfrac{m}{n}$ is in **simplest-form** if m and n have no common factors.

Definition 1.4.4 (Rational Numbers). Let x be a real number. The number x is said to be **rational** if there exists integers m and n, with n non-zero, such that $x = \dfrac{m}{n}$.

Axiom 1.3. After a finite number of cancellations, any fraction $\dfrac{m}{n}$ can be written in simplest form.

`Prove that` $\sqrt{2}$ `is irrational.`

***Proof* (7).**

Assume for the sake of contradiction that $\sqrt{2}$ is rational.

Then, by definition, $\exists p, q \in \mathbb{Z}$, with $q \neq 0$ *s.t.* $\sqrt{2} = \dfrac{p}{q}$. We can assume that p and q have no common factors.

Note that $\sqrt{2}q = p$, squaring both sides, we see that $2q^2 = p^2$.

Since $q^2 \in \mathbb{Z}$, by definition, p^2 is even. From Proof (4), if p^2 is even, then p must be even. By definition, $\exists k \in \mathbb{Z}$ *s.t.* $p = 2k$.

Plugging this back to our equation: $2q^2 = (2k)^2 = 4k^2$, $q^2 = 2k^2$.

Since $k^2 \in \mathbb{Z}$, we see that q^2 is even. Again, from Proof (4), if q^2 is even, then q is even.

⋇ This contradicts the fact that p, q had no common factors.

So, $\sqrt{2}$ must, in fact, be irrational.

■

`Let` $0 < \alpha < 1$. `Prove that` $\sqrt{\alpha} > \alpha$.

***Proof* (8).**

We will use the Proof by Contradiction.

Suppose $0 < \alpha < 1$. Assume for the sake of contradiction that $\sqrt{\alpha} < \alpha$.

Since $\alpha, \sqrt{\alpha} > 0$ and $f(x) = x^2$ is an increasing function for positive x, squaring both sides, we have $\alpha \leq \alpha^2$.

So, $\alpha^2 - \alpha \geq 0$. Then $\alpha(\alpha - 1) \geq 0$.

$$\begin{cases} \alpha \geq 0 \\ \alpha - 1 \geq 0 \end{cases} \qquad \text{or} \begin{cases} \alpha \leq 0 \\ \alpha - 1 \leq 0 \end{cases}$$

We have $\alpha \geq 1 \quad$ or$\alpha \leq 0$.

⋇ This contradicts with the fact that $0 < \alpha < 1$.

So, $\sqrt{\alpha} > \alpha$.

■

Another way to prove is by contrapositive.

***Proof* (9).**

We will prove the contrapositive: If $\sqrt{\alpha} \leq \alpha$, then $\alpha \leq 0$ or $\alpha \geq 1$.

Suppose $\alpha \in \mathbb{R}$ and $\sqrt{\alpha} \leq \alpha$.

As $\sqrt{\alpha}$ exists, $\alpha \geq 0$.
Since $\alpha, \sqrt{\alpha} \geq 0$ and $f(x) = x^2$ is an increasing function for $x \geq 0$: $\alpha \leq \alpha^2$.
Then $\alpha^2 - \alpha \geq 0$, $\alpha(\alpha - 1) \geq 0$.
Since $\alpha \geq 0$, $\alpha - 1$ must also be greater than 0.
So, we have $\begin{cases} \alpha \geq 0 \\ \alpha - 1 \geq 0 \end{cases}$
Hence, we have $\alpha \geq 1$.

■

`Prove that there are no integer solutions to the equation` $x^2 = 4y + 2$.

***Proof* (10).**
Assume for the sake of contradiction that $\exists x, y \in \mathbb{Z}$ *s.t.* $x^2 = 4y + 2$.

$$x^2 = 4y + 2 = 2(2y + 1)$$

Since $2y + 1 \in \mathbb{Z}$, x^2 is an even number, and thus x is also an even number.
Then, $\exists k \in \mathbb{Z}$ *s.t.* $x = 2k$.

$$\begin{aligned} \therefore x^2 = (2k)^2 = 4k^2 &= 4y + 1 \\ 2k^2 &= 2y + 1 \end{aligned}$$

$\boxed{\text{Method } 1}$
Since $k^2 \in \mathbb{Z}$, $2k^2$ is even. Since $y \in \mathbb{Z}$, $2y + 1$ is odd.
Then the equation indicates an even number equals to an odd number.

⋇ This contradicts with the fact that even numbers cannot equal to an odd number.

$\boxed{\text{Method } 2}$

$$\begin{aligned} 2k^2 &= 2y + 1 \\ 2(k^2 - y) &= 1 \\ k^2 - y &= \frac{1}{2} \end{aligned}$$

⋇ This contradicts with the fact that since $k, y \in \mathbb{Z}$, so does $k^2 - y$.

So, our assumption is wrong. There is no integer solutions for $x^2 = 4y + 2$.

■

Definition 1.4.5 (Prime Numbers). A prime number is a natural number that is only divisible by 1 and itself.

Example 1.4.1. 2, 3, 5, 7, 11, 13, 17, 19, ...

```
Prove there are infinitely many primes
```

Remark. This is a very revealing proof by contradiction. It reveals a way to find a larger prime based on the primes that we already have.

***Proof* (11).**

Assume for the sake of contradiction that there are exactly n primes, where $n \in \mathbb{N}$. Let's list them as $p_1,\ p_2,\ p_3, \cdots, p_n$, where $p_1 = 2,\ p_2 = 3, \cdots, p_n$ is the largest prime.

Consider the number $q = \underbrace{p_1 p_2 p_3 \cdots p_n}_{\text{product}} + 1$. Then notice that since $p_i \geq 2 \forall i$, we have $q > p_n$.

Since $q > p_n$, q cannot be prime, based on our assumption.

So, $\exists p_k \in \{p_1,\ p_2,\ \cdots,\ p_n\}$ *s.t.* $p_k \mid q$. Then, by definition, $q = cp_k$ *f.s.* $c \in \mathbb{Z}$.

So,

$$c = \frac{q}{p_k} = \frac{p_1 p_2 p_3 \cdots p_n + 1}{p_k} = \frac{p_1 p_2 p_3 \cdots p_n}{p_k} + \frac{1}{p_k}$$

Then,

$$\frac{1}{p_k} = c - \frac{p_1 p_2 p_3 \cdots p_n}{p_k}$$

Note that since p_k is one prime among $p_1, \cdots p_n$, $\dfrac{p_1 p_2 p_3 \cdots p_n}{p_k} \in \mathbb{Z}$.

Since $c \in \mathbb{Z}$, then $\dfrac{1}{p_k} \in \mathbb{Z}$.

⋇ This is a contradiction because $p_k \nmid 1$ as $p_k \geq 2 \forall k$.

So, there must be infinitely many primes

■

If $n \in \mathbb{Z}$, then $5n^2 + 3n + 7$ is odd.

***Proof* (12).**

Suppose $n \in \mathbb{Z}$.

$\boxed{\text{Case 1}}$ If n is even. Then $\exists k \in \mathbb{Z}$ *s.t.* $n = 2k$.

$$5n^2 + 3n + 7 = 5(2k)^2 + 3(2k) + 7 = 20k^2 + 6k + 7 = 2(10k^2 + 3k + 3) + 1$$

Since $10k^2 + 3k + 3 \in \mathbb{Z}$, $5n^2 + 3n + 7$ is odd.

$\boxed{\text{Case } 2}$ If n is odd. Then $\exists k \in \mathbb{Z}$ *s.t.* $n = 2k + 1$.

$$5n^2 + 3n + 7 = 5(2k+!)^2 + 3(2k + 1) + 7 = 20k^2 + 26k + 15 = 2(10k^2 + 13k + 7) + 1$$

Since $10k^2 + 13k + 7 \in \mathbb{Z}$, $5n^2 + 3n + 7$ is odd.

■

Definition 1.4.6 (Equivalence). The prove $P \iff Q$ (equivalence or biconditional, "if and only if" or " *iff* ") we must prove the forward implication $P \implies Q$ and the backward implication $Q \implies P$.

`Prove that for` $x, y \in \mathbb{R}$, $(x + y)^2 = x^2 + y^2$ `if and only if` $x = 0$ `or` $y = 0$.

Proof **(13).**

($\Rightarrow$): We will show that $(x + y)^2 = x^2 + y^2 \implies x = 0$ or $y = 0$.

Suppose $x, y \in \mathbb{Z}$ *s.t.* $(x + y)^2 = x^2 + y^2$. So we have

$$x^2 + y^2 + 2xy = x^2 + y^2$$

That is, $2xy = 0$ or $xy = 0$. So $x = 0$ or $y = 0$ □

($\Leftarrow$): We will show that $x = 0$ or $y = 0 \implies (x + y)^2 = x^2 + y^2$.

Notice that the equation retains symmetry.

Suppose $x, y \in \mathbb{R}$. Assume $x = 0$ or $y = 0$.

Without loss of generality (WLOG), we may assume that $x = 0$.

Then,

$$(x + y)^2 = (0 + y)^2 = y^2$$
$$x^2 + y^2 = 0^2 + y^2 = y^2$$

So, $(x + y)^2 = x^2 + y^2$.

■

2 Sets

2.1 Sets and Subsets

Definition 2.1.1 (Set, Element). A **set** is a collection of objects, called **elements**. A set is typically denoted by a capital letter and a set's elements are listed in $\{\}$.

Example 2.1.1.
- $A = \{1, 2, 3\}$
- $F = \{1, 2, 3, 5, 8, 13, ...\}$ the set of Fibonacci numbers
- $\mathbb{N} = \{1, 2, 3, 4, 5, ...\}$ the **natural numbers** (Does not include 0.)
- $\varnothing$ the **empty set**
- $M = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$
- $\{$Ludacris, T.I., Killer Mike, Big Boi, Andre 3000, Latto$\}$

Definition 2.1.2. We write $a \in A$ (a is in A) to indicate a is an element of A and $a \notin A$ (a is not in A) to indicate a is not an element of A.

Definition 2.1.3 (Cardinality). The **cardinality** of a set A is the number of elements in A and is denoted $|A|$.

Remark. When listing the elements of a set, each element is listed only once (for example, $\{1, 1\}$ is not a set) and the order of the elements doesn't matter ($\{1, 2\} = \{2, 1\}$).

Notation 2.1 (Set Builder Notation). We can use the set builder notation to describe a general property of the elements.

Example 2.1.2. $\{2n \mid n \in \mathbb{N}\}$ is "the set of elements of the form $2n$ such that n is a natural number."

Remark. We use $2\mathbb{Z}$ to represent $\{2k \mid k \in \mathbb{Z}\}$. Similarly, we have $5\mathbb{Z}, 10\mathbb{Z}, ...$

Definition 2.1.4 (Subset). We say a set A is a **subset** of a set B, denoted $A \subseteq B$, if every element of A is an element of B (If $a \in A$, then $a \in B$). We write $A \nsubseteq B$ if A is not a subset of B ($\neg(A \subseteq B) = a \in A$ and $a \notin B$.)

- $A \subseteq B$: "A is a subset of B"
- $B \supseteq A$: "B contains A"

- $A \subsetneq B$: "A is a proper subset of B" or "A is a subset of B, but $A \neq B$."

Remark. By definition, $A \subseteq A$ (every set is a subset of itself).

Remark. $\varnothing$ is a subset of any set.

Theorem 2.1.1. A set of n elements has 2^n subsets.

***Proof* (1).**

For every element in the set, they have two options: in the subset or out the subset. Hence, there will be in total 2^n subsets.

■

Definition 2.1.5. Building New Sets From Old

- **Intersection**: $A \cap B = \{x \mid x \in \ A \text{ and } x \in B\}$
- **Union**: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- **Difference**: $A - B = A \backslash B = \{x \mid x \in A \text{ and } x \notin B\}$

Theorem 2.1.2.

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

$$A - B \neq B - A$$

Definition 2.1.6 (Universal Set). A **universal set** of A is a set within which A exists.

Definition 2.1.7 (Complement). The **complement** of A in its universal set U is $\overline{A} = A^C = U - A$.

Theorem 2.1.3.

$$A \cup \overline{A} = U$$

$$A \cap \overline{A} = \varnothing$$

Specially, if U is finite, $|U| = |A| + |\overline{A}|$

Theorem 2.1.4 (De Morgan's Laws).

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Set Theoretic Proofs: Prove $a \in A$: To prove an element belongs to a set, we must prove the element satisfies the required properties of the set.

`Prove that` $14 \in \{4a + 7b : \ a, b \in \mathbb{Z}\}$

***Proof* (2).**

Note that $14 = 4(0) + 7(2)$

Since $0, 2 \in \mathbb{Z}$, by definition, $14 \in \{4a + 7b \mid a, b \in \mathbb{Z}\}$.

■

Theorem 2.1.5. Proving $A \subseteq B$: To prove $A \subseteq B$, we must prove that if $a \in A$, then $a \in B$. We can do so using any technique:

- Direct proof: Suppose $a \in A$...... Therefore, $a \in B$.
- Contrapositive proof: Suppose $a \notin B$...... Therefore, $a \notin A$.
- Contradiction: Suppose $a \in A$. Assume for the sake of contradiction that $a \in B$......⁂

`Prove that for all` $m, n \in \mathbb{Z}$`, we have` $\{x \in \mathbb{Z} : mn \mid x\} \subseteq \{x \in \mathbb{Z} : m \mid n\} \cap \{x \in \mathbb{Z} : n \mid x\}$.

***Proof* (3).**

Let $x \in \mathbb{Z}$ ***s.t.*** $x \in \{x \in \mathbb{Z} : mn \mid x\}$. Then, $mn \mid x$, and so $x = kmn$ ***f.s.*** $k \in \mathbb{Z}$.

So, $x = m(kn)$, and since $kn \in \mathbb{Z}$, $x \in \{x \in \mathbb{Z} : m \mid x\}$.

Also, $x = n(km)$, and since $km \in \mathbb{Z}$, $x \in \{x \in \mathbb{Z} : n \mid x\}$.

By definition of intersection, $x \in \{x \in \mathbb{Z} : m \mid x\} \cap \{x \in \mathbb{Z} : n \mid x\}$.

■

Definition 2.1.8 (Equal). We say two sets A and B are **equal**, denoted $A = B$, if they have the same elements.

Remark. $A = B \iff A \subseteq B$ and $B \subseteq A$

`If` A`,` B`, and` C `are sets, prove` $(A \cap B) - C = (A - C) \cap (B - C)$

***Proof* (4).**

($\subseteq$): $(A \cap B) - C \subseteq (A - C) \cap (B - C)$

Suppose $x \in (A \cap B) - C$ (WTS: $x \in (A - C) \cap (B - C)$)

By definition of intersection, $x \in A$ and $x \in B$

By definition of difference, $x \notin C$.

Since $x \in A$ and $x \notin C$, $x \in A - C$. Similarly, since $x \in B$ and $x \notin C$, $x \in B - C$.

By definition of intersection, $x \in (A - C) \cap (B - C)$ □

($\supseteq$): $(A - C) \cap (B - C) \subseteq (A \cap B) - C$

Let $x \in (A - C) \cap (B - C)$.
By definition of intersection, $x \in A - C$ and $x \in B - C$.
By definition of difference, $x \in A$, $x \notin C$ and $x \in B$, $x \notin C$.
Since $x \in A$ and $x \in B$, $x \in (A \cap B)$
Further since $x \notin C$, by definition of difference, $x \in (A \cap B) - C$.

■

`If` A `and` B `are sets, prove that` $A \subseteq B$ `if and only if` $\overline{B} \subseteq \overline{A}$.

***Proof* (5).**

($\Rightarrow$): $A \subseteq B \implies \overline{B} \subseteq \overline{A}$.

Suppose $A \subseteq B$. [WTS: $\overline{B} \subseteq \overline{A}$]
Suppose $x \in \overline{B}$. [WTS: $x \in \overline{A}$]
Since $x \in \overline{B}$, then $x \notin B$. Note that since $A \subseteq B$, by definition, if $x \in A$, then $x \in B$.
By contrapositive, $x \notin B \implies x \notin A$
Since $x \notin B$, by assumption, $x \notin A$. That is $x \in \overline{A}$. □

($\Leftarrow$): $\overline{B} \subseteq \overline{A} \implies A \subseteq B$.

Suppose $\overline{B} \subseteq \overline{A}$ [WTS: $A \subseteq B$]
Suppose $x \in A$ [WTS: $x \in B$]
Since $x \in A$, then $x \notin \overline{A}$. Since$\overline{B} \subseteq \overline{A}$, by contrapositive,

$$x \notin \overline{A} \implies x \in \overline{B}.$$

Since $x \notin \overline{A}$, by assumption, $x \notin \overline{B}$. That is, $x \in B$.

■

2.2 Combining Sets

`Prove that` $A \subseteq B$ `if and only if` $A - B = \varnothing$.

***Proof* (1).**

($\Rightarrow$): WTS: $A \subseteq B \implies A - B = \varnothing$.

Let A, B be sets. Suppose $A \subseteq B$.
Assume for the sake of contradiction that $A - B \neq \varnothing$. Then $x \in A - B$.
By definition of set difference, $x \in A$ and $x \notin B$.

$\divideontimes$ This contradicts with the fact that $A \subseteq B$, and thus

$$A \subseteq B \implies A - B = \varnothing. \qquad \square$$

($\Leftarrow$): WTS: $A - B = \varnothing \implies A \subseteq B$.

$\boxed{\text{Method 1}}$ We will prove the contrapositive: $A \nsubseteq B \implies A - B \neq \varnothing$.

Suppose $A \nsubseteq B$. Then $\exists x \in A$ *s.t.* $x \notin B$.

By definition of set difference, $x \in A - B$, so $A - B \neq \varnothing$ as desired.

$\boxed{\text{Method 2}}$ Direct proof.

Suppose $A - B = \varnothing$.

By definition of set difference, $\forall x \in A, x \in B$.

By definition of subsets, $A \subseteq B$.

■

Definition 2.2.1 (Power set). The **power set** of A, denoted $\mathcal{P}(A)$, is the set of all subset of A. Therefore, $|\mathcal{P}(A)| = 2^n$ when $|A| = n$.

Theorem 2.2.1.

$$x \in \mathcal{P}(A) \iff x \subseteq A$$

`Let` A, B `be sets. Prove that` $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$

***Proof* (2).**

Let A, B be sets.

Suppose $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$ [Notation: Capital X because X is a set!]

Then, by definition of union, $X \in \mathcal{P}(A)$ or $X \in \mathcal{P}(B)$.

WLOG, suppose $X \in \mathcal{P}(A)$. By definition of the power set, $X \subseteq A$.

Since $A \subseteq A \cup B$, if $X \subseteq A$, then $X \subseteq A \cup B$.

By definition of the power set, $X \in \mathcal{P}(A \cup B)$.

Hence, $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

■

Definition 2.2.2 (Cartesian Product). The **Cartesian product** of two sets A and B, denoted $A \times B$, is the set of ordered pairs (or tuples) (a, b) where $a \in A$ and $b \in B$, so

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Remark. In general, $A \times B \neq B \times A$.

Theorem 2.2.2.

$$|A \times B| = |A| \times |B|$$

```
Let A, B, and C be sets.  Prove that if B ⊆ C, then A × B ⊆ A × C.
```

***Proof* (3).**

Let A, B, and C be sets. Suppose $B \subseteq C$. WTS: $A \times B \subseteq A \times C$.

Let $x \in A \times B$. Then $x = (a, b)$ where $a \in A$ and $b \in B$.

Since $B \subseteq C$, if $b \in B$, then $b \in C$.

So, $x = (a, b) \in A \times C$.

Hence, $A \times B \subseteq A \times C$.

■

Definition 2.2.3 (Cartesian Power). For a set A and $n \in \mathbb{N}$, the **Cartesian power** A^n is the set

$$A^n = A \times A \times A \times \cdots \times A = \{\underbrace{(a_1, a_2, a_3, \cdots, a_n)}_{n\text{-tuple}} \mid a_1, a_2, a_3, \cdots, a_n \in A\}.$$

Example 2.2.1. Most common example: the Euclidean space: $\mathbb{R}^n$.

2.3 Collection of Sets

Definition 2.3.1 (Indexed Sets).

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n = \{x \mid x \in A_i \textit{ f.s. } i\}$$

and

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n = \{x \mid x \in A_i \ \forall i\}$$

This notation also works when we don't have specified stopping point: $\bigcup_{i=1} A_i = \{x \mid x \in A_i \textit{ f.s. } i\}$ and $\bigcap_{i=1} A_i = \{x \mid x \in A_i \ \forall i\}$

Example 2.3.1. Given the indexed sets, compute the unions and intersections:

1. $A_i = [0, i)$ for $i = 1, \cdots, n$

   (a) $\bigcup_{i=1}^{n} A_i = [0, n)$

   ***Proof* (1).**

   Claim: $\bigcup_{i=1}^{n} A_i = [0, 1)$.

   ($\subseteq$) Let $x \in \bigcup_{i=1}^{n} A_i$. So, $\exists k \in \{1, 2, \cdots, n\}$ *s.t.* $x \in A_k$.

By definition of A_k, $x \in [0, k)$.

That is, $0 \geq x < k$.

Since $k \leq n$, this inequality gives us that $0 \leq x < k \leq n$.

So, $x \in [0, n)$ as desired. $\square$

($\supseteq$) Let $x \in [0, n)$.

Since $x \in [0, n)$, note that $[0, n) = A_n$.

So $x \in A_n$.

By definition of union, $x \in \bigcup_{i=1}^{n} A_i$

Remark. To show $x \in \bigcup_{i=1}^{n} A_i$, find a specific i that works.

$\blacksquare$

(b) $\bigcap_{i=1}^{n} A_i = [0, 1)$

***Proof* (2).**

Claim: $\bigcap_{i=1}^{n} A_i = [0, 1)$.

($\subseteq$) Let $x \in \bigcap_{i=1}^{n} A_i$.

So, $x \in A_i \; \forall i \in \{1, 2, \cdots, n\}$.

Specifically, $x \in A_1 = [0, 1)$. $\square$

($\supseteq$) Let $x \in [0, 1)$ [WTS: $x \in A \; \forall i \in \{1, 2, c \ldots, n\}$]

Let $k \in \{1, 2, \cdots, n\}$. We will show $x \in A_k$.

We know that $k \geq 1$ and $0 \leq x < 1$, so we have $0 \leq x < 1 \leq k$.

That is, $x \in [0, k)$.

So, $x \in A_k$

Since $k \in \{1, 2, \cdots, n\}$ was ***arbitrary***, we have shown that $x \in A_k \; \forall k \in \{1, 2, \cdots, n\}$.

So, $x \in \bigcap_{i=1}^{n} A_i$.

$\blacksquare$

Remark. If $i \in \mathbb{Z}$, we have

(a) $\bigcup_{i=1}^{\infty} A_i = [0, \infty)$

(b) $\bigcap_{i=1}^{\infty} A_i = [0, 1)$

2. $A_i = [i-1, i]$ for $i = 1, \cdots, n$

   (a) $\displaystyle\bigcup_{i=1}^{n} A_i = [0, n]$

   (b) $\displaystyle\bigcap_{i=1}^{n} A_i = \begin{cases} [0,1] & \text{if } n = 1 \\ \{1\} & \text{if } n = 2 \\ \varnothing & \text{if } n > 2 \end{cases}$

   Remark. If $i \in \{1, 2, \cdots\} = \mathbb{N}$, then we have

   (a) $\displaystyle\bigcup_{i=1}^{\infty} A_i = [0, \infty)$

   (b) $\displaystyle\bigcap_{i=1}^{n} A_i = \varnothing$

`Prove or disprove: If` A, B, `and` C `are sets, and` $A \times C = B \times C$, `then` $A = B$.

***Proof* (3).**

If $C = \varnothing$, then $A \times C = \varnothing = B \times C$

But A and B could be dramatically different.

■

`Prove or disprove: If` A, B, `and` C `are sets, with` $C \neq \varnothing$, `and` $A \times C = B \times C$, `then` $A = B$.

***Proof* (4).**

Suppose A, B, and C are sets.

Suppose $\exists a, c \in \mathbb{Z}$ ***s.t.*** $(a, c) = A \times C$

By definition of Cartesian product, $a \in A$ and $c \in C$.

Suppose $\exists b, c \in \mathbb{Z}$ ***s.t.*** $(b, c) \in B \times C$

Similarly, $b \in B$.

Suppose $A \times C = B \times C$. Then $A \times C \subseteq B \times C$ and $B \times C \subseteq A \times C$

($\subseteq$) WTS: $A \subseteq B$.

  If $A \times B \subseteq B \times C$, we have $(a, c) \in B \times C$

  Then, $a \in B$.

  Since $a \in A$, we know $A \subseteq B$.

($\supseteq$) WTS: $B \subseteq A$

  Similarly, since $B \times C \subseteq A \times C$, we have $(b, c) \in A \times C$.

  Then, $b \in A$.

Since $b \in B$, we see $B \subseteq A$.
By definition of set equality, $A = B$.

■

Definition 2.3.2 (Partition). Let A be sets. A **partition** of A is a subset of the power set of A, $\mathscr{P} \subseteq \mathcal{P}(A)$, satisfying the following three properties:

1. If $X \in \mathscr{P}$, then $X \neq \varnothing \longrightarrow$ non-empty subsets of A
2. $\displaystyle\bigcup_{X \in \mathscr{P}} X = A$
3. If $X, Y \in \mathscr{P}$ and $X \neq Y$, then $X \cap Y = \varnothing \longrightarrow$ mutually disjoint.

Example 2.3.2. Let $A = \mathbb{Z}$ and consider the collection of subsets:

$$\mathscr{P} = \{\{\cdots, -4, -2, 0, 2, 4, \cdots\}, \{\cdots, -3, -1, 1, 3, 5, \cdots\}\}$$

Remark. Divisibility partisions $\mathbb{Z}$: Pick $n \in \mathbb{N},\ n > 1$. For every $m \in \mathbb{Z}$:

$$m = kn + r,$$

where r is the remainder, and $r = \{0, 1, 2, 3, \cdots, n-1\}$

Theorem 2.3.1 (The Pigeonhole Principle - Idea). You have m pigeons and n pigeonholes (boxes for the pigeons). If $m > n$, then there will be at least one pigeonhole with more than pigeon. If $m < n$, then you will have at least one empty pigeonhole.

Theorem 2.3.2 (The Pigeonhole Principle). Let $A_1, A_2, \cdots, A_n$ be a collection of finite mutually disjoint sets. Let $A = \displaystyle\bigcup_{i=1}^{n} A_i$. if $|A| = k$ and $k > n$, then $|A_i| \geq 2$ *f.s.* i.

`Given any 11 integers, there is a pair of numbers whose difference is divisible by 10.`

Proof **(5).**
Let $\{a_1, a_2, \cdots, 1_n\}$ be 11 integers.
Then, $a_i = 10k_i + r_i$ for $i \in \{1, \cdots, 11\}$, for $r_i = \{0, 1, 2, \cdots, 9\}$, and for $k_i \in \mathbb{Z}$.
Then, there are 10 possibilities for r_i, but there are 11 integers.
So, by the Pigeonhole Principle, at least 2 integers in our list must leave the same remainder on division by 10.

Say:

$$a_l = 10k_l + r \qquad \text{and} \qquad a_m = 10k_m + r \qquad \text{where} r \in \{0, 1, 2, \cdots, 9\}$$

Then,

$$\begin{aligned} a_l - a_m &= 10k_l + r - (10k_m + r) \\ &= 10k_l - 10k_m = 10(k_l - k_m) \end{aligned}$$

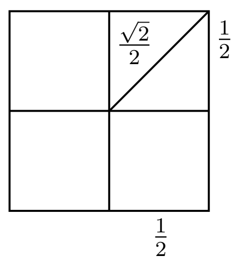
Since $k_l - k_m \in \mathbb{Z}$, we say that $10 \mid (a_l - a_m)$.

■

`Given` 5 `points in a unit square, there are two points within` $\dfrac{\sqrt{2}}{2}$ `of each other.`

Remark. Think carefully about the partition that gives the result!

Proof **(6).**
Split the square horizontally and vertically by picking the midpoints of each side.



This divides our square into 4 bins.

Since there are 4 bins and 5 points, by the Pigeonhole Principle, at least 2 must be in the same bin.

Further note that in each bin, the largest distance between two points is $\dfrac{\sqrt{2}}{2}$.

Hence, there are 2 points with $\dfrac{\sqrt{2}}{2}$ of each other.

■

3 The Integers

3.1 Axioms and Basic Properties

Definition 3.1.1 (Binary Operations). Binary operations of integers combine two integers to get another integer.

Axiom 3.1 (Integers). The **integers**, which we denoted by $\mathbb{Z}$, is a set, together with a nonempty subset $\mathbb{P}(\mathbb{Z}^+/\mathbb{N}) \subset \mathbb{Z}$ (which we call the **positive** integers), and two binary operations addition and multiplication, denoted by $+$ and $\cdot$, satisfying the following properties:

- (**Commutativity**) For all integers a, b, we have

$$a + b = b + a \qquad \text{and} \qquad a \cdot b = b \cdot a.$$

- (**Associativity**) For all integers a, b, c, we have

$$a + (b + c) = (a + b) + c \qquad \text{and} \qquad a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

- (**Distributivity**) For all integers a, b, c, we have

$$(a + b) \cdot c = a \cdot c + b \cdot c.$$

- (**Identity**) There exists integers 0 (**additive identity**) and 1 (**multiplicative identity**), such that for all integers a, we have

$$a + 0 = a \qquad \text{and} \qquad a \cdot 1 = a.$$

- (**Additive inverses**) For any integer a, there exists an integer $-a$ such that

$$a + (-a) = 0.$$

- (**Closure for $\mathbb{Z}^+$**) If a, b are positive integers, then $a + b$ and $a \cdot b$ are positive integers.

- (**Trichotomy**) For every integer a, exactly one of the following three possibilities hold: either a is a positive integer, or $a = 0$, or $-a$ is a positive integer.

- (**Well-ordering**) Every nonempty subset of the positive integers has a smallest element.

Remark. We do not talk about the multiplicative inverse because for almost all integers, there does not exist an integer multiplicative inverse. Most of the multiplicative inverse belongs to $\mathbb{Q}$, and only 1 and -1 are two integers that have integer multiplicative inverses.

Claim 3.1.1. Let $a, b, c \in \mathbb{Z}$. If $a + b = a + c$, then $b = c$.

Proof **(1).**
Since $a \in \mathbb{Z}$, $\exists(-a) \in \mathbb{Z}$ *s.t.* $a + (-a) = 0$ by *additive inverse.*
Adding $(-a)$ to the left side:

$$\begin{aligned}(-a) + (a + b) &= ((-a) + a) + b \qquad \big[\textit{associativity}\big] \\ &= 0 + b \qquad \big[\textit{additive inverse}\big] \\ &= b \qquad \big[\textit{additive identity}\big]\end{aligned}$$

Similarly, we can show $(-a) + (a + c) = c$
So, we show that $b = c$.

■

Claim 3.1.2. Let $a \in \mathbb{Z}$. Then, $a \cdot 0 = 0 \cdot a = 0$.

Proof **(2).**
Note by *commutativity*, $a \cdot 0 = 0 \cdot a$. So, it's enough to show $a \cdot 0 = 0$.
Notice that

$$\begin{aligned}a \cdot 0 + a \cdot 0 &= a(0 + 0) \qquad \big[\textit{distributivity}\big] \\ &= a \cdot 0 \qquad \big[\textit{additive identity}\big] \\ &= a \cdot 0 + 0 \qquad \big[\textit{additive identity}\big]\end{aligned}$$

So, $a \cdot 0 + a \cdot 0 = a \cdot 0 + 0$. By Claim 3.1.1, we see that $a \cdot 0 = 0$.

■

Claim 3.1.3. Let $a, b \in \mathbb{Z}$. Then $(-a)b = a(-b) = -ab$.

Remark. The three expressions are different: $(-a)b$ means the additive inverse of a and multiplying that by b; and $-ab$ is the additive inverse of the product of a and b.

Proof **(3).**
In this proof, we will show that $(-a)b = -ab$, and the other side will be left as an exercise.
WTS: $(-a)b$ is the additive inverse of ab. That is, WTS: $(-a)b + ab = 0$.

Notice that

$$\begin{aligned}(-a)b + ab &= ((-a)+a)b \quad [\textit{distributivity}]\\ &= 0 \cdot b \quad [\textit{additive inverse}]\\ &= 0 \quad [\text{From Claim 3.1.2}]\end{aligned}$$

So, $(-a)b$ is the additive inverse of ab. That is,

$$(-a)b = -ab.$$

■

Claim 3.1.4. If $a, b \in \mathbb{Z}$, then $(-a)(-b) = ab$.

Remark. We leave the proof of this claim as an exercise.

`Let` $x \in \mathbb{Z}$ `and` $x \neq 0$. `Then` $x^2 = x \cdot x \in \mathbb{Z}^+$.

***Proof* (4).**

Suppose $x \in \mathbb{Z}$ and $x \neq 0$. Since $x \neq 0$, by *trichotomy*, $x \in \mathbb{Z}^+$ or $-x \in \mathbb{Z}^+$

$\boxed{\text{Case } 1}$ Suppose $x \in \mathbb{Z}^+$. Then, $x^2 = x \cdot x \in \mathbb{Z}^+$ by *closure for* $\mathbb{Z}^+$.

$\boxed{\text{Case } 2}$ Suppose $-x \in \mathbb{Z}^+$. Then, $(-x)(-x) \in \mathbb{Z}^+$ by *closure for* $\mathbb{Z}^+$.

But $(-x)(-x) = x \cdot x = x^2$ by Claim 3.1.4. So, $x^2 \in \mathbb{Z}^+$.

■

Notation 3.1.

$$a - b = a + (-b)$$

$$ab = (a) \cdot (b)$$

Definition 3.1.2. Let $x, y \in \mathbb{Z}$. We say $x < y$ if $y - x \in \mathbb{Z}^+$.

Remark. We can use the axioms of closure and trichotomy to establish some well-known facts about inequalities.

Claim 3.1.5. Let $a, b, c \in \mathbb{Z}$. Then, exactly one of the following holds: $a = b$, $a < b$, or $b < a$.

***Proof* (5).**

Let $a, b \in \mathbb{Z}$

As before, by *additive inverse*, $\exists (-b) \in \mathbb{Z}$ *s.t.* $b + (-b) = 0$. Then $a - b \in \mathbb{Z}$.

By *trichotomy*, there are 3 cases to consider:

$\boxed{\text{Case } 1}$ $a - b = 0$, so then adding b on both sides, we get $a = b$.
$\boxed{\text{Case } 2}$ $a - b \in \mathbb{Z}^+$, then by Definition 3.1.2, $a > b$.
$\boxed{\text{Case } 3}$ $-(a - b) \in \mathbb{Z}^+$. But

$$\begin{aligned} -(a-b) &= -a - (-b) && \left[\textit{distributivity}\right] \\ &= -a + b \\ &= b - a. \end{aligned}$$

So, by Definition 3.1.2, $b > a$.

■

Claim 3.1.6. Let $a \in \mathbb{Z}$. If $a > 0$, then $-a < 0$, and if $a < 0$, then $-a > 0$.

Claim 3.1.7. Let $a, b \in \mathbb{Z}$. If $a > 0$ and $b > 0$, then $ab > 0$.

Claim 3.1.8. Let $a, b, c \in \mathbb{Z}$. If $a < b$ and $b < c$, then $a < c$.

Claim 3.1.9. Let $a, b, c \in \mathbb{Z}$. If $a < b$, then $a + c < b + c$.

Theorem 3.1.1 (Well-Ordering Principle). Every non-empty subset of the positive integers has a smallest element. (If $X \subseteq \mathbb{Z}^+$ and $X \neq \varnothing$, then $\exists x_0 \in X$ *s.t.* $\forall a \in X$ with $a \neq x_0$, we have $a - x_0 \in \mathbb{Z}^+$.)

Theorem 3.1.2. There is no integer x such that $0 < x < 1$.

Proof **(6).**
Assume for the sake of contradiction that $\exists x \in \mathbb{Z}$ *s.t.* $0 < x < 1$.
Let $S = \{n \in \mathbb{Z} \mid 0 < n < 1\}$.
By our assumption, $x \in S$, so $S \neq \varnothing$.
Note that, by definition, $S \subseteq \mathbb{Z}^+$.
By the Well-Ordering Principle (Theorem 3.1.1), S has a smallest element, say s_0.
We know that $0 < s_0 < 1$. Then $1 - s_0 \in \mathbb{Z}^+$.
Since $s_0, 1 - s_0 \in \mathbb{Z}^+$, there product $s_0(1 - s_0) \in \mathbb{Z}^+$ $\left[\textit{closure for } \mathbb{Z}^+\right]$
That is, $s_0 - s_0^2 \in \mathbb{Z}^+$.
By Definition 3.1.2, we have $s_0 > S_0^2$.
As $x_0 \neq 0$, $s_0^2 \in \mathbb{Z}^+$ $\left[\text{Problem } 3.1\right]$
So, $0 < s_0^2 < s_0 < 1$.

⁎ This is a contradiction because s_0 is assumed to be the smallest element of S.

So, $\nexists x \in \mathbb{Z}^+$ *s.t.* $0 < x < 1$, or there are no integers between 0 and 1.

■

3.2 Induction

Corollary 3.1 (Introduction to Mathematical Induction). The most powerful consequence of the well-ordering principle (Theorem 3.1.1) is the technique of mathematical induction. The central idea of mathematical induction is that

To prove a statement that depends on an integer n, denoted $P(n)$, it is enough to prove it about a smaller integer m, $P(m)$, and then show that since $m < n$, $P(m) \implies P(n)$.

Theorem 3.2.1 (First Principle of Mathematical Induction). Let $P(n)$ be a statement about the positive integer n. Suppose that

1. $P(1)$ is true; and
2. $\forall k \in \mathbb{Z}$ *s.t.* $k \geq 2,\ P(k-1) \implies P(k)$.

Then, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

Remark. Equivalently, we can write the Condition 2 as $\forall k \geq 1,\ P(k) \implies P(k+1)$.

The following is a prove of Theorem 3.2.1

`If` $P(1)$ `is true, and` $\forall k \geq 2,\ \ P(k-1) \Rightarrow P(k),$ `then` $P(n)$ `is true` $\forall n \in \mathbb{N}$.

***Proof* (1).**

Let $P(n)$ be a statement about positive integers.

Suppose $P(1)$ is true and $\forall k \geq 2,\ P(k-1) \implies P(k)$.

Assume for the sake of contradiction that $\exists a \in \mathbb{Z}^+$ *s.t.* $P(a)$ is false.

Let $F = \{x \in \mathbb{Z}^+ \mid P(x) \text{ is false}\}$. Note that $F \subseteq \mathbb{Z}^+$. Further by assumption, $a \in F$, and so $F \neq \varnothing$.

So, by WOP (Well-Ordering Principle, Theorem 3.1.1), $\exists f_0 \in F$ that is the smallest element of F. [WTS: $F = \varnothing$]

Note that since $P(1)$ is true, then $1 \notin F$. Consider numbers $f_0 - 1 \in \mathbb{Z}$.

Since 1 is the smallest positive integer (a corollary of Theorem 3.1.2), and $f_0 \in \mathbb{Z}^+$, we have $f_0 > 1$. So, $f_0 - 1 \in \mathbb{Z}^+$.

Since f_0 is the smallest element of F and $f_0 - 1 \in \mathbb{Z}^+$. Then $P(f_0 - 1)$ is true. (Because $f_0 - 1 < f$ cannot be in the set of F, by our assumption).

By assumption, $P(f_0)$ is true because $F(f_0 - 1) \implies P(f_0)$.

$$\divideontimes \text{ But } f_0 \in F \text{ so } P(f_0) \text{ is false.}$$

So, it must be that $\forall n \in \mathbb{Z}^+,\ P(n)$ is true.

■

Remark. This proof strategy is also called proof by **the smallest counterexample.**

`The sum of the first` n `natural numbers is` $\dfrac{n(n+1)}{2}$.

***Proof* (2).**

Let $P(n)$ be the statement

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

$\boxed{\text{Base Case}}$ Consider $P(1) : 1 = \dfrac{1(1+1)}{2}$.

The right side is $\dfrac{1(2)}{2} = 1$. So, the statement is clearly true.

$\boxed{\text{Inductive Steps}}$ Assume that for some $k \geq 1,\ P(k)$ is true. That is,

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \qquad ①$$

WTS: $P(k+1)$ is true. That is, we WTS:

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)(k+1+1)}{2} = \frac{(k+1)(k+2)}{2}$$

Adding $(k+1)$ to both sides of equation ①:

$$\begin{aligned}
(1 + 2 + 3 + \cdots + k) + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\
&= (k+1)\left(\frac{k}{2} + 1\right) \\
&= (k+1)\left(\frac{k}{2} + \frac{2}{2}\right) \\
&= (k+1)\frac{(k+2)}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

So, $(k+1)$ is true.

We've shown that $P(1)$ is true and $\forall k \geq 1,\ P(k) \implies P(k+1)$.
So, by Principle of Mathematical Induction (Theorem 3.2.1), $P(n)$ is true $\forall n \in \mathbb{Z}^+$.

■

Remark (The Role of the Base Case). Mathematical induction only works if you can "get the ball rolling." However, our first step in mathematical induction **need not to be** $n = 1$.

Definition 3.2.1 (Factorial). We define factorial of a positive integer n as

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$$

Notably, we define the factorial of 0 as 1. That is,

$$0! = 1.$$

`Prove that if` $n \in \mathbb{Z}$ `and` $n > 4$, `then` $n! > 2^n$.

***Proof* (3).**
Let $P(n)$ be the statement that $n! > 2^n$.
$\boxed{\text{Base Case}}$ WTS: $P(4)$ is true.

$$4! = (1)(2)(3)(4) = 24, \qquad 2^4 = 16.$$

Since $24 > 16,\ P(4)$ is true.
$\boxed{\text{Inductive Step}}$ Suppose *f.s.* $k \geq 4,\ P(k)$ is true. That is, $k! > 2^k$ ①
WTS: $P(k+1)$ is true. That is, $(k+1)! > 2^{k+1}$.
Multiplying both sides of ① by $(k+1)$: $(k!)(k+1) > 2^k(k+1)$.
So, we have $(k+1)! > 2^k(k+1) > 2^k(2) = 2^{k+1}$ since $k \geq 4$.
That is, $(k+1)! > 2^{k+1}$.
Since $P(4)$ is true and $\forall k \geq 4,\ P(k) \implies P(k+1)$, by Mathematical Induction Principle (Theorem 3.2.1), $P(n)$ is true $\forall n \geq 4$.

■

Remark (Generalizing Theorem). Often the role of induction is to generalize theorems from cases with few elements to an arbitrary but finite number of elements.

Example 3.2.1. Let $n \in \mathbb{Z}^+$. The following theorems can be proven using the Mathematical Induction Principle (Theorem 3.2.1).:

1. Suppose $a_1, a_2, \cdots, a_n$ are all even integers. Then $\displaystyle\sum_{i=1}^{n} a_i$ is even.

2. Suppose A and $B_1, B_2, \cdots, B_n$ are all sets. Then

$$A \cap \left(\bigcup_{i=1}^{n} B_i \right) = \bigcup_{i=1}^{n} (A \cap B_i)$$

3. **Theorem 3.2.2.** Let A be a finite set with n elements. Then, $\mathcal{P}(A)$ has 2^n elements.

Definition 3.2.2. Fibonacci Numbers

$$F_1 = 1 \qquad F_2 = 1 \qquad F_n = F_{n-1} + F_{n-2} \texttt{ for } n \geq 3$$

Example 3.2.2. The *Fibonacci numbers* are described as follows:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \cdots$$

`Prove that the Fibonacci sequence satisfies:`

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$$

`for all` n.

***Proof* (4).**

Let $P(n)$ be the statement that $F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$

$\boxed{\text{Base Case}}$ $P(1): \ F_1 = 1, \ F_{2(1)} = F_2 = 1$, So, $P(1)$ is true.

$\boxed{\text{Inductive Step}}$ Suppose *f.s.* $k \geq 1, \ P(k)$ is true. That is,

$$F_1 + F_3 + \cdots + F_{2k-1} = F_{2k} \qquad ①$$

WTS: $P(k+1)$ is true. That is, $F_1 + F_3 + \cdots + F_{2k-1} + F_{2k+1} = F_{2k+2}$.

Add F_{2k+1} to both sides of equation ①:

$$F_1 + F_3 + \cdots + F_{2k-1} + F_{2k+1} = F_{2k} + F_{2k+1} = F_{2k+2}$$

So, $P(k+1)$ is true.

Since $P(1)$ is true and $\forall k \geq 1, \ P(k) \implies P(k+1)$, by Mathematical Induction Principle (Theorem 3.2.1), $P(n)$ is true $\forall n \in \mathbb{Z}^+$.

■

`Prove that if` $n \in \mathbb{Z}$ `and` $n \geq 0$, `then` $6 \mid 7^n - 1$.

***Proof* (5).**

Let $P(n)$ be the statement "$6 \mid 7^n - 1$."

$\boxed{\text{Base Case}}$ WTS: $P(0)$ is true. That is, $6 \mid 7^0 - 1$.

Since $7^0 - 1 = 1 - 1 = 0$ and $6 \mid 0$, then $6 \mid 7^0 - 1$.

That is, $P(0)$ is true.

$\boxed{\text{Inductive Step}}$ Suppose for some $k \geq 0$, $P(k)$ is true. That is, $6 \mid 7^k - 1$.

In other words, $\exists l \in \mathbb{Z}$ *s.t.* $7^k - 1 = 6l$ or $7^k = 6l + 1$.

WTS: $P(k+1)$ is true. That is, $6 \mid 7^{k+1} - 1$

Note that

$$\begin{aligned} 7^{k+1} - 1 = 7^k \cdot 7 - 1 &= 7(6l+1) - 1 \\ &= 42l + 7 - 1 \\ &= 42l + 6 \\ &= 6(7l+1). \end{aligned}$$

Since $7l + 1 \in \mathbb{Z}$, by definition of divides, $6 \mid 7^{k+1} - 1$.

Hence, $P(k) \implies P(k+1)$.

Since we've proven $P(0)$ is true and for $k \geq 0$ $P(k) \implies P(k+1)$, by Principle of Mathematical Induction (Theorem 3.2.1), for all $n \in \mathbb{Z}$ and $n \geq 0$, $P(n)$ is true.

■

Theorem 3.2.3 (Second Principle of Mathematical Induction). Let $P(n)$ be a statement about the positive integer n. Suppose that

1. $P(1)$ is true.
2. $\forall k \in \mathbb{Z}$ such that $k \geq 2$, if $P(i)$ is true for all $1 \leq i \leq k-1$, then $P(k)$ is true.

Then, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

Remark. The phrasing "$P(i)$ is true for all $1 \leq i \leq k-1$, then $P(k)$ is true" can also be phrased as "$\forall k \geq 1$, $P(i)$ is true $\forall i \quad 1 \leq i \leq k$. Then, $P(k+1)$ is true.

Remark. The meaning of this phrasing, "$P(i)$ is true for all $1 \leq i \leq k-1$" is that $P(1)$ is true, $P(2)$ is true, $P(3)$ is true, $\cdots$, $P(k-2)$ is true, and $P(k-1)$ is true.

Remark. The Second Principle of Mathematical Induction (Theorem 3.2.3) is logically equivalent to the First Principle of Mathematical Induction (Theorem 3.2.1). However, in some cases, we have to use Theorem 3.2.3 in our proofs.

Remark. When using the Second Principle of Mathematical Induction, think about base cases! For example, if we need to show for $k-1 \geq 0$ $(k \geq 1)$ $P(k+1)$ is true, then our base cases are $P(0)$ and $P(1)$.

`Given` $n \in \mathbb{N}$`, define a recursive function` f `as follows:`

$$f(0) = 1 \qquad f(1) = 3 \qquad f(n) = 2f(n-1) - f(n-2) \quad \forall n \geq 2.$$

`Prove that for all` $n \geq 0$`,` $f(n) = 2n+1$`.`

***Proof* (6).**

Let $P(n)$ be the statement "$f(n) = 2n+1$."

$\boxed{\text{Base Cases}}$ WTS: $P(0)$ and $P(1)$ are true.

$P(0)$ is true: $\left[\text{WTS: } f(0) = 2(0)+1\right]$

Note that $2(0)+1 = 0+1 = 1$ and $f(0) = 1$ by definition. That is, $P(0)$ is true.

$P(1)$ is true: $\left[\text{WTS:} f(1) = 2(1)+1\right]$

Since $f(1) = 3$ by definition and $2(1)+1 = 3$, $P(1)$ is true.

$\boxed{\text{Inductive Steps}}$ Let $k \geq \mathbf{1}$ be *s.t.* $\forall i \quad 0 \leq i \leq k,\ P(i)$ is true. – *inductive hypothesis*

WTS: $P(k+1)$ is true.

Since $k \geq 1$, $k+1 \geq 2$. So, by definition,

$$\begin{aligned} f(k+1) &= 2f(k+1-1) - f(k+1-2) \\ &= 2f(k) - f(k-1) \end{aligned}$$

Since $k-1 \geq 0$, by inductive hypothesis, $P(k)$ and $P(k-1)$ are true.

So, $f(k-1) = 2(k-1)+1$ and $f(k) = 2k+1$.

Substituting, we have

$$\begin{aligned} f(k+1) &= 2(2k+1) - [2(k-1)+1] \\ &= 4k+2-2k+2-1 \\ &= 2k+3 \\ &= 2k+2+1 \\ &= 2(k+1)+1. \end{aligned}$$

Hence, by Mathematical Induction (Theorem 3.2.3), $P(n)$ is true $\forall n \geq 0$.

■

4 Equivalence Relations

4.1 Equivalence Relations

Definition 4.1.1 (Relation). A **relation** R on a set A is a subset of $A \times A = \{(a, b) \mid a, b \in A.\}$

Notation 4.1. If two elements $a, b \in A$ are related, we write $(a, b) \in R$ or aRb. If a and b are not related, we write $(a, b) \notin A$ or $a \not R b$.

Extension. More generally, if A and B are sets, then a relation R from A to B is a subset of $A \times B$.

Example 4.1.1. Describe the following relations by listing the elements of the relation

1. Let $A = \{1, 2, 3, 4, 5\}$ and $R_<$ is the relation "strictly less than."

   Answer. aRb if $a < b$.

$$R_< = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}.$$

□

2. Let $A = \{1, 2, 3, 4, 5\}$ and $R_|$ is the relation "divides."

   Answer.

$$R_| = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 4), (3, 3), (4, 4), (5, 5)\}.$$

□

3. Let $A = \mathbb{R}$ and $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x = y\}$.

   Answer. This relation describes the line of $y = x$ in a Cartesian plane. □

   Remark. Any subset of $A \times A$ is a relation. A relation does not have to have a actual meaning such as "strictly less than" or "divides."

Definition 4.1.2 (Reflexive, Symmetric, Antisymmetric, Transitive). A relation R on a set A is

- **Reflexive** if $(a, a) \in R \quad \forall a \in A$.

  ***Proof* (1).**

  Suppose $a \in A$. We'll show that $(a, a) \in R$.

  ■

  ***Disproof* (2).**

  $\exists a \in A$ ***s.t.*** $(a, a) \notin R$.

  ■

- **Symmetric** if $\forall a, b \in A$, if $(a, b) \in R$,then $(b, a) \in R$.

  Proof **(3).**

  Suppose $a, b \in A$. Suppose $(a, b) \in R$.

  We'll show that $(b, a) \in R$.

  ■

  Disproof **(4).**

  $\exists a, b \in A$ ***s.t.*** $(a, b) \in R$ and $(b, a) \notin R$.

  ■

- **Antisymmetric** if $\forall a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$.

  Proof **(5).**

  Suppose $a, b \in A$. Suppose (a, b) and $(b, a) \in R$.

  We'll show that $a = b$.

  ■

  Disproof **(6).**

  $\exists a, b \in A$ ***s.t.*** (a, b) and $(b, a) \in R$ and $a \neq b$.

  ■

- **Transitive** if $\forall a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

  Proof **(7).**

  Suppose $a, b, c \in A$. Suppose (a, b) and $(b, c) \in R$

  We'll show that $(a, c) \in R$.

  ■

Example 4.1.2. Show if the following relations are reflexive, symmetric, antisymmetric, or transitive.

1. Suppose $A \subseteq \mathbb{Z}$ or $\mathbb{N}$. Consider $R_<$.

   - **Reflexive**: False.

     Counterexample. $(1, 1) \notin R_<$. □

   - **Symmetric**: False.

     Counterexample. $(1, 2) \in R_<$, but $(2, 1) \notin R_<$. □

- **Antisymmetric**: True.

  ***Proof* (8).**

  Impossible for (a, b) and $(b, a) \in R$. So the assumption part of the implication is wrong, which means the implication overall is true.

  ■

- **Transitive**: True

  ***Proof* (9).**

  Suppose $a, b, c \in A$ *s.t.* $a < b,\ b < a$

  Then, $a < c$. That is, $(a, c) \in R$.

  ■

2. Suppose $A \subseteq \mathbb{Z}$ or $\mathbb{N}$. Consider $R_|$.

- **Reflexive**: True

  ***Proof* (10).**

  Recall that $a \mid b$ if $b = ak$ *f.s.* $k \in \mathbb{Z}$.

  So, when $k = 1$, $b = a$. That is, $a \mid a \forall a \in Z$.

  That is, $(a, a) \in R$.

  ■

- **Symmetric**: False.

  Counterexample. $(1, 2) \in R_|$, but $(2, 1) \notin R_|$. □

- **Antisymmetric**: True if $A = \mathbb{N}$; False if $A = \mathbb{Z}$.

  ***Proof* (11).**

  Suppose $a, b \in A$ *s.t.* $a \mid b$ and $b \mid a$.

  Then, $b = ak$ and $a = bl$ *f.s.* $k, l \in \mathbb{Z}$.

  So, $a = (ak)l = akl$.

$$
\begin{aligned}
a - akl &= 0 \\
a(1 - kl) &= 0 \\
a = 0 \qquad \text{or} \qquad kl &= 1 \\
&\ k = l = 1 \quad \text{or} \quad k = l = -1.
\end{aligned}
$$

  That is, $a = 0$ or $a = b$ or $a = -b$. So the relation is antisymmetric if $A = \mathbb{N}$ because then only the option $a = b$ makes sense. If $A = \mathbb{Z}$, all three options are possible.

  ■

- **Transitive**: True.

  ***Proof* (12).**

  $a \mid b \implies b = ak \qquad b \mid c \implies c = bl$

  $\implies c = (ak)l = a(kl) \implies a \mid c.$

  ■

3. Consider the relation $R = \{(x, y) \in R^2 \mid x^2 + y^2 = 1\}$.

- **Reflexive**: False.

  Counterexample. $(0, 0) \notin R$ because $0^2 + 0^2 \neq 1.$ □

- **Symmetric**: True

  ***Proof* (13).**

  Let $x, y \in \mathbb{R}$ *s.t.* xRy. Then, $x^2 + y^2 = 1.$

  Then, $y^2 + x^2 = 1$. That is, $yRx.$

  ■

- **Antisymmetric**: False.

  Counterexample. Consider $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \in R$, but $\frac{\sqrt{3}}{2} \neq \frac{1}{2}$

  Or, consider $(0, 1)$ and $(1, 0) \in R$, but $0 \neq 1.$ □

- **Transitive**: False.

  Counterexample. $(0, 1) \in R$ and $(1, 0) \in R$, but $(0, 0) \notin R.$ □

4. Consider the relation $R_{\leq} = \{(x, y) \in \mathbb{R}^2 \mid x \leq y\}$.

- **Reflexive**: True.

- **Symmetric**: False.

  Counterexample. $(1, 2) \in R$, but $(2, 1) \notin R.$ □

- **Antisymmetric**: True.

  ***Proof* (14).**

  If $a \leq b$ and $b \leq a$, then $a = b.$

  ■

- **Transitive**: True.

  ***Proof* (15).**

  If $a \leq b$ and $b \leq c$, then $a \leq c.$

  ■

5. Let S be the set of all finite subsets of $\mathbb{Z}$. Consider the relation R on S by ARB if $|A| = |B|$.

   - **Reflexive**: True.

     ***Proof* (16).**

     $|A| = |A|$.

     ■

   - **Symmetric**: True.

     ***Proof* (17).**

     If $|A| = |B|$, then $|B| = |A|$.

     ■

     .

   - **Antisymmetric**: False.

     Counterexample. If $A = \{1, 2\}$ and $B = \{2, 3\}$, then $|A| = |B|$, but $A \neq B$. □

   - **Transitive**: True.

     ***Proof* (18).**

     If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

     ■

Definition 4.1.3 (Equivalence Relation). A relation R on a set A is an **equivalence relation** on A if it is reflexive, symmetric, and transitive.

Notation 4.2. When R is an equivalence relation, it is common to write $a \sim b$ (read as "a is equivalent to b") instead of aRb.

Remark. Equivalence relations are a way to generalize the idea of "equality."

Prove that $R_=$ on the set $\mathbb{R}$ is an equivalence relation.

***Proof* (19).**

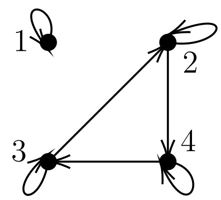
- **Reflexive**: $a = a \quad \forall a \in \mathbb{R}$
- **Symmetric**: $a = b \implies b = a \quad \forall a, b \in \mathbb{R}$.
- **Transitive**: $a = b,\ b = c \implies a = c \quad \forall a, b, c \in \mathbb{R}$.

■

Remark. We can use graphs and arrows to represent a relation.

- **Reflexive**: Every vertex needs a self loop.
- **Symmetric**: Every edge that is not a loop should look like $\Longleftrightarrow$
- **Transitive**: We should have something like vector addition for every three vertexes.

Example 4.1.3. The relation $R = \{(1,1),(2,2),(2,4),(3,2),(3,3),(4,3),(4,4)\}$ on the set $A = \{1,2,3,4)\}$ as the follwoing graph:



From the graph we know that R is reflexive, but R is not symmetric or transitive.

Definition 4.1.4 (Congruence Modulo). Given a natural number $n \neq 1$, we defined a relation R on $\mathbb{Z}$, called **congruence mod** n, by aRb if $a-b = nk$ for some $k \in \mathbb{Z}$. That is, aRb if $n \mid (a-b)$.

Notation 4.3. We write $a \equiv b \mod n$ if a is congruent to b modulo n.

Remark. Two integers a and b are congruent modulo n if they give the same remainder upon division by n.

```
Let n ∈ ℕ. Prove that the relation
```

$$\equiv \mod n$$

```
on the set ℤ is an equivalence relation.
```

Proof **(20).**

Let $n \in \mathbb{N}$.

- **Reflexive**: Let $a \in \mathbb{Z}$. $\quad$ [WTS: $a \equiv a \mod n$]

  We can see that $n \mid (a-a) = 0$. So, $a \equiv a \mod n$. ☐

- **Symmetric**: Let $a, b \in \mathbb{Z}$ *s.t.* $a \equiv b \mod n$. $\quad$ [WTS: $b \equiv a \mod n$]

  By definition 4.1.4, $n \mid (a-b)$ or $a - b = nk$ *f.s.* $k \in \mathbb{Z}$.

  Multiplying by (-1), we get that

$$-(a-b) = -nk$$
$$b - a = n(-k).$$

Since $(-k) \in \mathbb{Z}$, then $n \mid (b-a)$. That is, $b \equiv a \mod n$.

Hence, R is symmetric. □

- **Transitive**: Let $a, b, c \in \mathbb{Z}$ *s.t.* $a \equiv b \mod n$ and $b \equiv c \mod n$.

  WTS: $a \equiv c \mod n$.

  Since $n \mid (a-b)$ and $n \mid (b-c)$, then $a-b = nk$ and $b-c = nl$ *f.s.* $k, l \in \mathbb{Z}$.

  Adding the two equations, we get

$$\begin{aligned}(a - \cancel{b}) + (\cancel{b} - c) &= nk + nl \\ a - c &= nk + nl = n(k+l)\end{aligned}$$

  Since $(k+l) \in \mathbb{Z}$, $n \mid (a-c)$. So, $a \equiv c \mod n$.

  Hence, R is transitive.

■

Definition 4.1.5 (Equivalence Class). Let R be an equivalence relation on a set A. Given $a \in A$, the **equivalence class containing** a is the subset

$$\{a \in A \mid x \sim a\}$$

Note that $\{a \in A \mid x \sim a\} = \{x \in A \mid a \sim x\}$ because x is symmetric.

Notation 4.4. We denote the equivalence class containing a by $[a]$.

Remark. Elements of the same equivalence class are said to be *equivalent*.

Proposition 4.1. Let R be an equivalence relation on a set A. Suppose $a, b \in A$. Then, $[a] = [b]$ if and only if aRb.

Proof **(21).**
Let R be an equivalence relation on a set A. Suppose $a, b \in A$.
($\Rightarrow$): Suppose $[a] = [b]$. [WTS: aRb]
Recall $[a] = \{x \in A \mid x \sim a\}$ and $[b] = \{x \in A \mid x \sim b\}$.
Note that since R is **reflexive**, aRa. So, by definition, $a \in [a]$.
Since $[a] = [b]$, we have $a \in [b]$.
By definition 4.1.5, aRb as desired.

($\Leftarrow$): Suppose aRb. $\big[$WTS: $[a] = [b]$, $[a] \subseteq [b]$ and $[b] \subseteq [a]\big]$

($\subseteq$): Suppose $x \in [a]$ $\big[$WTS: $x \in [b]\big]$

Since $x \in [a]$, by definition, xRa.

Since xRa and aRb, by **transitivity** of R, xRb.

So, $x \in [b]$. Hence, $[a] \subseteq [b]$.

($\supseteq$): Suppose $x \in [b]$ $\big[x \in [a]\big]$

Since $x \in [b]$, by definition, xRb.

Since aRb, by **symmetry**, bRa.

Since xRb and bRa, by **transitivity**, xRa.

So, $x \in [a]$. That is, $[b] \subseteq [a]$.

So, $[a] = [b]$.

■

Proposition 4.2. Let R be an equivalence relation on a set A. Then the set

$$\{[a] \mid a \in A\}$$

of equivalence classes of R forms a partition of A.

***Proof* (22).**

By Definition 2.3.2, we know that a partition of set A is a collection of non-empty subsets of A, union of all of which is A, and mutually disjoint.

By definition, if $a \in A$, $\{[a] \mid a \in A\} = \{x \in A \mid xRa\}$.

So, $\{[a] \mid a \in A\} \subseteq A$. Since R is reflexive, aRa, so $a \in [a]$.

So, this collection is indeed a collection of non-empty subsets of A. □

WTS: $\bigcup\limits_{a \in A} \{[a] \mid a \in A\} = A$.

($\subseteq$): Let $x \in \bigcup\limits_{a \in A} \{[a] \mid a \in A\}$.

By definition of union, $x \in \{[a] \mid a \in A\}$ *f.s.* $a \in A$.

But $\{[a] \mid a \in A\} \subseteq A$. So, $x \in A$.

($\supseteq$): Let $a \in A$. So, aRa and $a \in [a]$.

So, $a \in \{[a] \mid a \in A\}$. Then, $a \in \bigcup\limits_{a \in A} \{[a] \mid a \in A\}$. □

WTS: if $[a] \neq [b]$, then $[a] \cap [b] = \varnothing$.

We will prove by contrapositive: if $[a] \cap [b] \neq \varnothing$, then $[a] = [b]$.

Suppose $[a] \cap [b] \neq \varnothing$. Let $x \in [a] \cap [b]$.

So, $x \in [a]$ and $x \in [b]$. That means, xRa and xRb.

If $xRa,$ since R is symmetric, aRx.
Since R is transitive, since aRx and $xRb,$ we know $aRb.$
By Proposition 4.1, $[a] = [b]$.

■

Proposition 4.3. Let $\mathscr{P}$ be a partition of a nonempty set A. Define a relation R on A by aRb if a and b are in the same element of the partition. Then R is an equivalence relation on A.

***Proof* (23).**
The proof of Proposition 4.3 is given in the textbook.

■

5 Functions

5.1 Definition and Basic Properties

Definition 5.1.1 (Function, Domain, Codomain). Let A and B be non-empty set. A **function**, f, from A to B is a rule that assigns to each element $a \in A$ one, and only one, element of B, b.

- We call A the **domain** of f.
- We call B the **codomain** of f.

Notation 5.1. We denote a function f from A to B as $f : A \to B$, and the elements by $b = f(a)$ if b was assigned to a.

Remark. The codomain is part of the data that makes a function. Indeed, to specify a function, we need three pieces of information: (1) the domain, (2) the codomain, and (3) the rule between them.

Definition 5.1.2 (Image of a Function). Let $f : A \to B$ be a function. The **image** of f, denoted $\operatorname{Im}\{f\}$, is the set of elements in the codomain which are obtained by applying the rule f to an element of the domain. That is,

- $\operatorname{Im}(f) = \{y \in B \mid \exists a \in A \textit{ s.t. } b = f(a)\}$
- More generally, if $X \subseteq A$, then, the **image of** X **under** f, denoted $f(X)$, is given by

$$f(X) = \{y \in B \mid \exists x \in Z \textit{ s.t. } b = f(x)\}.$$

Claim 5.1.1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 + 3$. Then $\operatorname{Im}(f) = [3, \infty)$.

***Proof* (1).**
($\subseteq$): Let $x \in \mathbb{R}$. Then $x^2 \geq 0$.
So, $x^2 + 3 \geq 0 + 3$.
So, $\operatorname{Im}(f) \subseteq [3, \infty)$. $\square$
($\supseteq$): We will show that $(-3, \infty) \subseteq \operatorname{Im}(f)$
Let $x \in [3, \infty)$. [WTS: $x \in \operatorname{Im}(f)$. That is, $\exists a \in \mathbb{R}$ *s.t.* $f(a) = x$.]
Choose $a = \sqrt{x-3}$. Since $x \geq 3,\ x - 3 \geq 0$. So, $a \in \mathbb{R}$.
We will show that $f(a) = x$:

$$f(a) = a^2 + 3 = \left(\sqrt{x-3}\right)^2 + 3 = x - 3 + 3 = x$$

So, $x \in \operatorname{Im}(f)$.

■

Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by

$$f(n) = \begin{cases} n+1, & n \texttt{ is even} \\ n-3, & n \texttt{ is odd} \end{cases}$$

Compute $f(X)$ where $X = \{n \in \mathbb{Z} \mid n = 4t+2 \ \texttt{f.s.} \ t \in \mathbb{Z}\}$.

Claim 5.1.2. $f(X) = \{m \in \mathbb{Z} \mid m = 4k+3$ ***f.s.*** $k \in \mathbb{Z}\} = B$.

Proof **(2).**
($\subseteq$): Let $y \in f(X)$. Then, $\exists n \in X$ ***s.t.*** $f(n) = y$.
By definition of X, $n = 4t+2$ ***f.s.*** $t \in \mathbb{Z}$.
So, $f(n) = y = n+1$ since n is even.
That is, $f(n) = y = 4t+2+1 = 4t+3$.
So, $y \in B$
($\supseteq$): Let $m \in B$. Then, $m = 4k+3$ ***f.s.*** $k \in \mathbb{Z}$. [WTS: $m \in f(X)$]
Note that since $k \in \mathbb{Z}$, $4k+2 \in X$. Choose $n = 4k+2$.
Then, $n \in X$. Further, $f(n) = n+1 = 4k+2+1 = 4k+3 = m$.
So, $m \in f(X)$.

■

Theorem 5.1.1 (Intermediate Value Theorem). Let f be a function whose domain and codomain are subsets of R. Let $a < b$ and suppose f is continuous on $[a,b]$. If y is any number in-between $f(a)$ and $f(b)$, then there exists a number c in $[a,b]$ such that $f(c) = y$.

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3 + x + 1$. Let $X = [-2,2]$. Then, $f(X) = [-9,11]$.

Proof **(3).**
($\subseteq$): First observe that $f'(x) = 3x^2 + 1 > 0$.
Since $f' > 0$, f is increasing. So, f is in particular increasing on $[-2,2]$.
Note that $f(-2) = (-2)^3 + (-2) + 1 = -8 - 2 + 1 = -9$.
Also, $f(2) = 2^3 + 2 + 1 = 11$.
Since f is increasing, $\forall x \in [-2,2]$, $-9 \leq f(x) \leq 11$.
So, $f(X) \subseteq [-9,11]$. □
($\supseteq$):Let $m \in [-9,11]$.
Since f is continuous on $[-2,2]$ and $f(-2) = 9$ and $f(2) = 11$.
By the intermediate value theorem (Theorem 5.1.1), $\exists c \in [-2,2]$ ***s.t.*** $f(c) = m$.
That is, $m \in f(X)$.

■

`Let` $f : \mathbb{R} \to \mathbb{R}$ `be given by` $f(x) = x^3 - 4x + 1$. `Let` $X = [-2, 2]$. `Find` $f(X)$.

Proof **(4).**

① Find $f'(x) = 3x^2 - 4$. So, f is not always increasing or decreasing.

② Find critical points: $f'(x) = 0,\ 3x^2 - 4 = 0,\ 3x^2 = 4,\ x^2 = \dfrac{4}{3},\ x = \pm\dfrac{2}{\sqrt{2}}$.

③ Plug in critical points into $f''(x)$: $f''(x) = 6x$.

- $f''\left(\dfrac{2}{\sqrt{3}}\right) = 6\left(\dfrac{2}{\sqrt{3}}\right) > 0 \longrightarrow$ minimum
- $f''\left(-\dfrac{2}{\sqrt{3}}\right) = 6\left(-\dfrac{2}{\sqrt{3}}\right) < 0 \longrightarrow$ maximum

- Largest output is at $-\dfrac{2}{\sqrt{3}}$ or at 2.

  $f(2) = 2^3 - 4(2) + 2 = 1$

  $f\left(-\dfrac{2}{\sqrt{3}}\right) = \left(-\dfrac{2}{\sqrt{3}}\right)^3 - 4\left(-\dfrac{2}{\sqrt{3}}\right) + 1 = \dfrac{8}{\sqrt{3}}\left(\dfrac{2}{3}\right) + 1 > 1$

- Smallest output is at -2 or $\dfrac{2}{\sqrt{3}}$.

  $f(-2) = (-2)^3 - 4(-2) + 2 = 1$

  $f\left(\dfrac{2}{\sqrt{3}}\right) = \left(\dfrac{2}{\sqrt{3}}\right)^3 - 4\left(\dfrac{2}{\sqrt{3}}\right) + 1 = \dfrac{8}{\sqrt{3}}\left(-\dfrac{2}{3}\right) + 1 < 1$

Therefore, $f([-2, 2]) = \left[1 - \dfrac{2}{3}\left(\dfrac{8}{\sqrt{3}}\right), 1 + \dfrac{2}{3}\left(\dfrac{8}{\sqrt{3}}\right)\right]$

■

Theorem 5.1.2. Let $f : A \to B$ be a function, and let X and Y be subsets of A. Then:

1. $f(X \cup Y) = f(X) \cup f(Y)$
2. $f(X \cap Y) \subseteq f(X) \cap f(Y)$

`We will prove Theorem 3.2.1(1) in this section. Theorem 3.2.1(2) is left as an exercise.`

Proof **(5).**

($\subseteq$): Let $b \in f(X \cup Y)$.

Then, $\exists a \in X \cup Y$ *s.t.* $f(a) = b$.

Since $a \in X$ and $f(a) = b$, by definition of image, $b \in f(X)$.
Then, $b \in f(X) \cup f(Y)$. $\square$
($\supseteq$): Suppose $b \in f(X) \cup f(Y)$.
Then, $b \in f(X)$ or $b \in f(Y)$.
WLOG, suppose $b \in f(X)$.
Then, $\exists a \in X$ *s.t.* $f(a) = b$.
Since $a \in X$, $a \in X \cup Y$. So, $b = f(a) \in f(X \cup Y)$.

$\blacksquare$

Remark. Give a function $f : A \to B$ and subsets $X, Y \subseteq A$, it is not true, in general, that $f(X \cap Y) = f(X) \cap f(Y)$.

Definition 5.1.3 (Inverse Image, or Pre-Image). Let $f : A \to B$ be a function, and let $W \subseteq B$. The **inverse image** of W with respect to f denoted $f^{-1}(W)$, is the set of elements in the domain which are sent, by applying the rule f, to an element of W in B. That is,

$$f^{-1}(W) = \{a \in A \mid f(a) \in W\}.$$

Remark.

1. $f(X) \subseteq B$ but $f^{-1}(W) \subseteq A$.
2. To show $y \in f(X)$, **find** an $a \in X$ *s.t.* $f(a) = y$.
3. To show $y \in f^{-1}(W)$, **show** $f(y) \in W$.
4. $y \in f^{-1}(W) \iff f(y) \in W$.
5. If $f(X) = f(Y)$, we cannot determine if $X = Y$.
6. $f^{-1}(B) = A$, but $f(A) \neq B$.
7. If $f^{-1}(X) = f^{-1}(Y)$, we cannot determine if $X = Y$.

`Suppose` $f : \mathbb{R} \to \mathbb{R}$ `is defined by` $f(x) = 3x + 1$.

Claim 5.1.3. $f^{-1}(W_2) = (1, \infty)$, where $W_2 = (4, \infty)$

***Proof* (6).**
($\subseteq$): Let $x \in f^{-1}(W_2)$.
Then, $f(x) \in W_2 = (4, \infty)$
So, $f(x) = 3x + 1 > 4$. So, $x > 1$.
That is, $x \in (1, \infty)$. $\square$

($\supseteq$): Let $x \in (1, \infty)$. $\left[\text{WTS: } x \in f^{-1}(W_2).\right]$

Since $x > 1$, so $3x > 3$ and $3x + 1 > 4$.

That is, $f(x) > 4$.

So, $f(x) \in W_2$, or $x \in f^{-1}(W_2)$.

$\blacksquare$

Claim 5.1.4. $f^{-1}(W_4) = X$, where $W_4 = \mathbb{Z}$ and $X = \left\{\dfrac{k-1}{3} \mid k \in \mathbb{Z}\right\}$.

***Proof* (7).**

($\subseteq$): Let $x \in f^{-1}(W_4)$.

So, $f(x) \in W_4$. That is, $f(x) \in \mathbb{Z}$.

Hence, $3x + 1 = k$ *f.s.* $k \in \mathbb{Z}$.

Therefore, $x = \dfrac{k-1}{3}$. So, $x \in X$. $\square$

($\supseteq$): Let $x \in X$.

Then, $x = \dfrac{k-1}{3}$ *f.s.* $k \in \mathbb{Z}$. $\left[\text{WTS: } x \in f^{-1}(W_4). \text{ WTS: } f(x) \in W_4\right]$

$f(x) = f\left(\dfrac{k-1}{3}\right) = 3\left(\dfrac{k-1}{3}\right) + 1 = k - 1 + 1 = k.$

Since $k \in \mathbb{Z}$, $f(x) \in \mathbb{Z} = W_4$.

So, $x \in f^{-1}(W_4)$.

$\blacksquare$

Theorem 5.1.3. Let $f : A \to B$ be a function, and let W and Z be subsets of B. Then:

1. $f^{-1}(W \cup Z) = f^{-1}(W) \cup f^{-1}(Z)$
2. $f^{-1}(W \cap Z) = f^{-1}(W) \cap f^{-1}(Z)$

`We will prove Theorem 3.1.3(1) in this section. Theorem 3.1.3(2) is left as an exercise.`

***Proof* (8).**

($\subseteq$): Let $y \in f^{-1}(W \cup Z)$. Then, $f(y) \in W \cup Z$.

By definition of union, $f(y) \in W$ or $f(y) \in Z$.

WLOG, suppose $f(y) \in W$. Then, $y \in f^{-1}(W)$.

But $f^{-1}(W) \subseteq f^{-1}(W) \cup f^{-1}(Z)$.

So, $y \in f^{-1}(W) \cup f^{-1}(Z)$. $\square$

($\supseteq$): Let $y \in f^{-1}(W) \cup f^{-1}(Z)$. By definition of union, $y \in f^{-1}(W)$ or $y \in f^{-1}(Z)$.

WLOG, suppose $y \in f^{-1}(W)$. Then, $f(y) \in W$.

But, $W \subseteq W \cup Z$. So, $f(y) \in W \cup Z$.

So, $y \in f^{-1}(W \cup Z)$.

■

5.2 Continuity of a Function

Definition 5.2.1 (Epsilon-Delta Definition of Continuity). Let A and B be open intervals of $\mathbb{R}$, and let $f : A \to B$ be a function. Let $a \in A$. We say f is **continuous at** a if the following statement is true:

$$\forall \varepsilon > 0,\ \exists \delta > 0 \textit{ s.t. } |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

Remark. The following statement can be also rephrased as "If $x \in (a - \delta, a + \delta)$, then $f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)$."

1. $|x - a| < \delta$ means the inputs are at most δ away from a.
2. $|f(x) - f(a)| < \varepsilon$ means the outputs are at most ε away from $f(a)$.

Remark. The idea of writing a proof: given $\varepsilon > 0$, FIND $\delta > 0$ that works.

`Use the formal definition of continuity (Definition 5.2.1) to show that` $f(x) = 2x + 1$ `is continuous at` $x = 3$.

***Proof* (1).**
Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{2} > 0$.
Suppose $x \in \mathbb{R}$ *s.t.* $|x - 3| < \delta$.
Then,

$$\begin{aligned} |f(x) - f(3)| &= |2x + 1 - 7| \\ &= |2x - 6| \\ &= |2(x - 3)| \\ &= 2|x - 3| < 2 \cdot \delta = 2 \cdot \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we've shown that

$$\forall \varepsilon > 0,\ \exists \delta > 0 \textit{ s.t. } |x - 3| < \delta \implies |f(x) - f(3)| < \varepsilon.$$

■

Use the formal definition of continuity (Definition 5.2.1) to show that $f(x) = 3 + 5x$ is continuous at $x = 4$.

***Proof* (2).**

Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{5} > 0$.

Suppose $x \in \mathbb{R}$ *s.t.* $|x-4| < \delta$.

Then,

$$\begin{aligned} |f(x) - f(4)| &= |3 + 5x - 23| \\ &= |5x - 20| \\ &= |5(x-4)| \\ &= 5|x-4| < 5 \cdot \delta = 5 \cdot \frac{\varepsilon}{5} = \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we've shown that

$$\forall \varepsilon > 0,\ \exists \delta > 0 \textit{ s.t. } |x-4| < \delta \implies |f(x) - f(4)| < \varepsilon.$$

■

Remark. In this example, δ ONLY depends on ε! So, the same δ will work when showing $f(x)$ is continuous at $x = 1$.

Use the formal definition of continuity (Definition 5.2.1) to show that $f(x) = x^2$ is continuous at $x = 0$.

***Proof* (3).**

Let $\varepsilon > 0$ be given. Choose $\delta = \sqrt{\varepsilon}$.

Note that since $\varepsilon > 0$, $\delta > 0$.

Suppose $x \in \mathbb{R}$ *s.t.* $|x - 0| < \delta$. That is, $|x| < \delta$.

Then,

$$\begin{aligned} |f(x) - f(0)| = |x^2 - 0| = |x^2| &= |x| \cdot |x| \\ &< \delta \cdot \delta = \left(\sqrt{\varepsilon}\right) \cdot \left(\sqrt{\varepsilon}\right) = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we've shown that

$$\forall \varepsilon > 0,\ \exists \delta = \sqrt{\varepsilon} > 0 \textit{ s.t. } |x - 0| < \delta \implies |f(x) - f(0)| < \varepsilon.$$

■

Claim 5.2.1. The same choice of δ will not work when proving $f(x) = x^2$ is continuous at $x = 1$ as well.

WTS: $\exists \varepsilon$ *s.t.* for $\delta = \sqrt{\varepsilon}$, $\exists x \in \mathbb{R}$ *s.t.* $|x-1| < \delta$ and $|f(x) - f(1)| \geq \varepsilon|$.

Proof **(4).**
Consider what happens when $\varepsilon = 1$. Then, $\delta = \sqrt{\varepsilon} = \sqrt{1} = 1$.
Want: $|f(x) - f(1)| < \varepsilon = 1$. That is, $|f(x) - 1| < 1$.
So, $-1 < f(x) - 1 < 1$, or $0 < f(x) < 2$.
When $\delta = 1$, we have $|x - 1| < \delta = 1$.
That is, $-1 < x - 1 < 1$, or $0 < x < 2$.
Note that $\forall x \in (\sqrt{2}, 2)$, $f(x) > 2$.
So, it is not true that $|f(x) - 1| < \varepsilon$.

∎

`Use the formal definition of continuity (Definition 5.2.1) to show that` $f(x) = x^2$ `is continuous at` $x = 1$.

WTS: $\forall \varepsilon > 0, \exists \delta > 0$ *s.t.* $|x - 1| < \delta \implies |f(x) - f(1)| < \varepsilon$.

Proof **(5).**
Let $\varepsilon > 0$ be given. Choose $\delta = \min\left\{1, \frac{\varepsilon}{3}\right\}$ $\left(\delta \text{ is the smaller of } 1 \text{ and } \frac{\varepsilon}{3}\right)$.
Note that $\delta \leq 1$ and $\delta \leq \frac{\varepsilon}{3}$.
Suppose $x \in \mathbb{R}$ *s.t.* $|x - 1| < \delta$.
Since $|x - 1| < \delta \leq 1$. So, $-1 < x - 1 < 1$, or $0 < x < 2$.
So, $1 < x + 1 < 3$. That is, $|x + 1| < 3$.
Then,

$$\begin{aligned}
|f(x) - f(1)| = |x^2 - 1| &= |(x-1)(x+1)| \\
&= |x-1||x+1| \\
&< 3|x-1| \\
&< 3 \cdot \delta \\
&= 3 \cdot \min\left\{1, \frac{\varepsilon}{3}\right\} \\
&\leq 3 \cdot \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

Since ε was arbitrary, we've proven that

$$\forall \varepsilon > 0, \exists \delta = \min\left\{1, \frac{\varepsilon}{3}\right\} \textit{ s.t. } |x - 1| < \delta \implies |f(x) - f(1)| < \varepsilon.$$

∎

Remark. For quadratic functions, δ changes with ε and choice of a.

Definition 5.2.2 (Discontinuity). The definition of discontinuity is, essentially, the negation of the definition of continuity (Definition 5.2.1):

$$\exists \varepsilon > 0 \textit{ s.t. } \forall \delta > 0,\ \exists x \textit{ s.t. } |x-a| < \delta \text{ but } |f(x) - f(a)| \geq \varepsilon.$$

Consider the Heavy-side Function

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}.$$

Show that H is discontinuous at $x = 0$.

***Proof* (6).**
Choose $\varepsilon = \frac{1}{2}$.
Then, we need $|H(x) - H(0)| < \frac{1}{2}$.
That is, we want $|H(x) - 1| < \frac{1}{2}$, or $-\frac{1}{2} < H(x) - 1 < \frac{1}{2}$.
So, we want $\frac{1}{2} < H(x) < \frac{3}{2}$.
Notice that $\forall x \in (-\delta, 0),\ H(x) = 0$.
So, $H(x) \notin \left(\frac{1}{2}, \frac{3}{2}\right)$.

■

5.3 Injection, Surjection, and Bijection

Definition 5.3.1 (Surjective, Injective, and Bijective Functions). A function $f : A \to B$ is

- **injective** or **one-to-one** if $\forall x, y \in A$, $f(x) = f(y)$ implies $x = y$.
- **surjective** or **onto** if $\forall b \in B, \exists a \in A$ ***s.t.*** $f(a) = b$
- **bijective** if f is both injective and surjective.

Remark. If f is a surjective function, then

$$\text{Codomain of } f = \operatorname{Im}(f).$$

Check if the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 3n + 2$ is injective and/or surjective.

Claim 5.3.1. f is injective but nor surjective.

***Proof* (1).**

- **Injective**: Suppose $n, m \in \mathbb{Z}$ ***s.t.*** $f(n) = f(m)$.

  Since $f(n) = f(m)$, we have $3n + 2 = 3m + 2$.

  Then, $3n = 3m$, or $n = m$.

  Since $f(n) = f(m) \implies n = m$, we have f is injective.

- **Surjective**: Choose $n = 4 \in \mathbb{Z}$.

  Suppose for the sake of contradiction, $\exists a \in \mathbb{Z}$ ***s.t.*** $f(a) = 4$.

  Then, $3a + 2 = 4$. So, $3a = 2$, or $a = \dfrac{2}{3}$.

  ※ This contradicts with the fact that $a \in \mathbb{Z}$.

  So, f is not surjective.

■

`Check if the function` $f : \mathbb{Z} \to \mathbb{Z}$ `defined by`

$$f(n) = \begin{cases} n - 1, & n \texttt{ odd} \\ n + 3, & n \texttt{ even} \end{cases}$$

`is injective and/or surjective.`

Claim 5.3.2. f is both injective and surjective.

***Proof* (2).**

- **Injective**: Suppose $n, m \in \mathbb{Z}$ ***s.t.*** $f(n) = f(m)$.

  $\boxed{\text{Case 1}}$ n and m both even.

  So, $n + 3 = m + 3$. So, $n = m$.

  $\boxed{\text{Case 2}}$ n and m both odd.

  So, $n - 1 = m - 1$. So, $n = m$.

  $\boxed{\text{Case 3}}$ n is odd and m is even.

  If n is odd, $f(n) = n - 1$ is even.

If m is even, $f(m) = m + 3$, which is odd.

So, $f(n) \neq f(m)$. That is, this case is impossible.

Since $f(n) = f(m) \implies n = m$. So, f is injective.

- **Surjective**: WTS: $\forall b \in \mathbb{Z}, \exists a \in \mathbb{Z}$ *s.t.* $f(a) = b$.

Let $b \in \mathbb{Z}$.

$\boxed{\text{Case } 1}$ b is odd.

Since b is odd, we know that $a = b - 3 \in \mathbb{Z}$ and a is even.

Then, $f(a) = a + 3 = (b - 3) + a = b$.

So, for $a = b - 3,\ f(a) = b$.

$\boxed{\text{Case } 2}$ b is even.

Since b is even, we know that $a = b + 1 \in \mathbb{Z}$ and a is odd.

Then, $f(a) = a - 1 = (b + 1) - 1 = b$.

So, for $a = b + 1$, we have $f(a) = b$.

■

Definition 5.3.2 (Identity Function). The identity function on a set A is the function

$$i_A : A \to A$$

defined to be $i_A(x) = x$ for all $x \in A$. Note that for any set A, the identity function i_A is bijective.

Remark (The Importance of Domain/Codomain). Suppose $f : A \to B$ is a function. The notions of injective and surjective are *very* sensitive to the domain A and codomain B. Even if two functions are given by the same "formula" f, the domain for one might be too big (*and thus "over-hit" certain elements*) or the codomain might be too big (*and thus "under-hit" certain elements*).

Definition 5.3.3 (Tuple Equality).

$$(a, b) = (c, d) \iff a = c \text{ and } b = d.$$

Consider $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $f(n, m) = nm$. Is it injective? Is it surjective?

Claim 5.3.3. It is not injective.

***Proof* (3).**
Note that $f(1,4) = 1 \times 4 = 4$ and $f(2,2) = 2 \times 2 = 4$.
However, $(1,4) \neq (2,2)$. So f is not injective.

■

Claim 5.3.4. It is surjective.

***Proof* (4).**
Let $k \in \mathbb{Z}$ [WTS: $\exists (a,b) \in \mathbb{Z} \times \mathbb{Z}$ ***s.t.*** $f(a,b) = k$]
Note that $(1,k) \in \mathbb{Z} \times \mathbb{Z}$. Also, $f(1,k) = (1)(k) = k$.
Since $\exists (1,k) \in \mathbb{Z} \times \mathbb{Z}$ ***s.t.*** $f(1,k) = k$, f is surjective.

■

`Prove that the function` $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{3\}$ `defined by`

$$f(x) = \frac{3x}{x-2}$$

`is bijective.`

***Proof* (5).**

- **Injective**: Let $x, y \in \mathbb{R} - \{2\}$ ***s.t.*** $f(x) = f(y)$ [WTS: $x = y$]

  Since $f(x) = f(y)$, we have

  $$\begin{aligned} \frac{3x}{x-2} &= \frac{3y}{y-2} \\ 3x(y-2) &= 3y(x-2) \\ 3xy - 6x &= 3xy - 6y \\ -6x &= -6y \\ x &= y \end{aligned}$$

  So, f is injective.

- **Surjective**: Let $y \in \mathbb{R} - \{3\}$ [WTS: $\exists x \in \mathbb{R} - \{2\}$ ***s.t.*** $f(x) = y$]

  Let's set $x = \dfrac{2y}{y-3}$.

  Note that since $y \neq 3,\ x \in \mathbb{R}$.

  Also, if $x = 2$, then $\dfrac{2y}{y-3} = 2$. So, $2y = 2y - 6$ or $0 = -6$. This is impossible.

So, $\dfrac{2y}{y-3} \in \mathbb{R} - \{2\}$. Now,

$$f(x) = f\left(\frac{2y}{y-3}\right) = \frac{3\left(\dfrac{2y}{y-3}\right)}{\left(\dfrac{2y}{y-3}\right) - 2} = \frac{6y}{2y - 2(y-3)} = y.$$

So, f is surjective.

Since f is both injective and surjective, f is bijective.

■

`Prove that the function` $g : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{R})$ `given by` $g(A) = A \cup \{\pi\}$ `is injective but not surjective.`

***Proof* (6).**

- **Injective**: Suppose $X, Y \in \mathcal{P}(\mathbb{Z})$ *s.t.* $g(X) = g(Y)$ [WTS: $X = Y$]

  Note that $g(X) = g(Y)$ means $X \cup \{\pi\} = Y \cup \{\pi\}$.

  Suppose $x \in X$. Since $x \in X$, we have $x \in X \cup \{\pi\}$

  So, by assumption, $x \in Y \cup \{\pi\}$.

  So, by definition of union, $x \in Y$, or $x \in \{\pi\}$.

  Note that $x \notin \{\pi\}$ because $x \in \mathbb{Z}$ and $x \neq \pi$.

  So, then $x \in Y$. So, $X \subseteq Y$

  Similarly, $Y \subseteq X$. So, $X = Y$.

  So, g is injective.

- **Not Surjective**: Note that $\mathbb{Z} \subseteq \mathbb{R}$. So, $\mathbb{Z} \in \mathcal{P}(\mathbb{R})$.

  Further, $\forall X \in \mathcal{P}(\mathbb{Z}),\ g(X) = X \cup \{\pi\}$.

  So, $\pi \in g(X)$. So, since $\pi \in \mathbb{Z}$.

  We've shown that $\forall X \in \mathcal{P}(\mathbb{Z}), g(X) \neq \mathbb{Z}$.

  So, g is not surjective.

■

5.4 Composition and Invertibility

Definition 5.4.1 (Composition of Functions). Let $f : A \to B$ and $g : B \to C$ be functions, The **composition** of f with g, denoted $g \circ f$ is the function $g \circ f : A \to C$ defined by $(g \circ f)(x) = g(f(x))$ for all $x \in A$.

Notation 5.2. Sometimes, we can also use the notation gf to denote the composition $g \circ f$.

Remark. The order of function composition matters, but parenthesis do not matter.

Definition 5.4.2 (Function Equality). Two functions f and g are equals, $f = g$, if $\forall x \in$ Domain of $f =$ Domain of g, $f(x) = g(x)$.

Let $f : A \to B$, $g : B \to C$, and $h : C \to D$ be functions. Show that function composition is *associative*. That is, show that $(hg)f = h(gf)$.

***Proof* (1).**
Note that both $(hg)f$ and $h(gf)$ are functions from $A \to D$.
Suppose $a \in A$ $\Big[$WTS: $\big[(hg)f\big](a) = \big[h(gf)\big](a)\Big]$
Consider

$$\big[(hg)f\big](a) = (hg)(f(a)) = h\big(g(f(a))\big)$$

$$\big[h(gf)\big](a) = h((gf)(a)) = h\big(g(f(a))\big)$$

So, we see that $\forall a \in A$, $\big[(hg)f\big](a) = \big[h(gf)\big](a)$.
So, $(hg)f = h(gf)$.

■

Lemma 5.1. Let $f : A \to B$ and $g : B \to C$ be functions. If f and g are injective, then $g \circ f$ is injective.

***Proof* (2).**
Let $f : A \to B$ and $g : B \to C$ be functions.
Let f and g be injective.
Suppose $x, y \in A$ *s.t.* $(g \circ f)(x) = (g \circ f)(y)$, so $g(f(x)) = g(f(y))$.
Since g is injective, $f(x) = f(y)$.
Since f is injective, $x = y$.

■

Lemma 5.2. Let $f : A \to B$ and $g : B \to C$ be functions. If f and g are surjective, then $g \circ f$ is surjective.

***Proof* (3).**
Let $f : A \to B$ and $g : B \to C$ be functions.
Let f and g are surjective.
Let $c \in C$ [WTS: $\exists a \in A$ *s.t.* $(g \circ f)(a) = c$]
Since g is surjective, $\exists b \in B$ *s.t.* $g(b) = c$.
Since f is surjective, $\exists a \in A$ *s.t.* $f(a) = b$.
So, $g(f(a)) = g(b) = c$. So, $(g \circ f)(a) = c$.
So, we found $a \in A$ *s.t.* $(g \circ f)(a) = C$.
So, $g \circ f$ is surjective.

■

Proposition 5.1. Let $f : A \to B$ and $g : B \to C$ be functions. If f and g are bijective, so is $g \circ f$.

Remark. $f \circ g$ is not always defined because we don't know if $C \subseteq A$.

***Proof* (4).**
Let $f : A \to B$ and $g : B \to C$ be functions.
Suppose f and g are bijective.
Since f and g are injective, by Lemma 5.1, $g \circ f$ is injective.
Since f and g are surjective, by Lemma 5.2, $g \circ f$ is surjective.
So, by definition of bijectivity, $g \circ f$ is bijective.

■

Definition 5.4.3 (Set of Functions). Let A and B be non-empty sets. The **set of all functions from** A to B is denoted $\mathbf{F}(A, B)$. In the special case that $A = B$, we denote the set of functions as $\mathbf{F}(A)$, instead of $\mathbf{F}(A, A)$.

Definition 5.4.4 (Formal Definition of Composition of Functions). Let A, B, and C be non-empty sets. Then composition of functions defines a function

$$\mathbf{F}(B, C) \times \mathbf{F}(A, B) \to \mathbf{F}(A, C)$$

given by $(g, f) \mapsto g \circ f$.

Remark. Note that $g \in \mathbf{F}(B, C)$ and $f \in \mathbf{F}(A, B)$.

Theorem 5.4.1. If A and B are finite and non-empty,

$$|\mathbf{F}(A, B)| = |B|^{|A|}.$$

Definition 5.4.5 (Invertible Functions). Let $f : A \to B$ be a function. We say f is **invertible** if there exists a function $g : B \to A$ such that both of the following are true:

- $g \circ f = i_A$
- $f \circ g = i_B$

Notation 5.3. We denote the inverse of f as f^{-1}. So, we have $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$, where i_A and i_B are identity functions on B and A.

Remark.

$$f^{-1} \neq \frac{1}{f}$$

`Let` $f : \mathbb{R}\backslash\{1\} \to \mathbb{R}\backslash\{-1\}$ `be given by` $f(x) = \dfrac{x}{1-x}$. `Using the function` $g : \mathbb{R}\backslash\{-1\} \to \mathbb{R}\backslash\{1\}$, `given by` $g(x) = \dfrac{x}{1+x}$, `show that` f `is invertible` Need to check: $g \circ f = i_A$ and $f \circ g = i_B$.

***Proof* (5).**
Let $x \in A$.

$$(g \circ f)(x) = g(f(x)) = \frac{f(x)}{1 + f(x)} = \frac{\frac{x}{1-x}}{1 + \frac{x}{1-x}} = \frac{x}{1 - x + x} = \frac{x}{1} = x.$$

So, $\forall x \in A,\ (g \circ f)(x) = x$. So, $g \circ f = i_A$.
Similarly, we can show $(f \circ g)(x) = x \quad \forall x \in B$.

■

Remark (How would we find g if it is not given?). Use scratch work to compute the formula of $g(x)$, then check the domain and codomain, and show $f \circ g = i_B$ and $g \circ f = i_A$.

Theorem 5.4.2. Let A and B be non-empty sets, and let $f \in \mathbf{F}(A, B)$. Then, f is invertible if and only if f is bijective.

Remark. Let $f : A \to B$ be a function.

- f is **bijective** means f preserves and captures the information between A and B.

- F is **invertible** means the action of f from A to B can be undone by using a function from B to A.

Theorem 5.4.2 is saying that the notions of **bijectivity** and **invertibility** are logically **equivalent**. So, a function f is either both bijective and invertible or it is neither.

`Prove Theorem 5.4.2: Let` A `and` B `be non-empty sets, and let` $f \in \mathbf{F}(A, B)$. `Then,` f `is invertible if and only if` f `is bijective.`

Proof **(6).**

($\Rightarrow$): Suppose f is invertible [WTS: f is bijective.]

That is, $\exists f^{-1} \in \mathbf{F}(B, A)$ *s.t.* $f \circ f^{-1} = i_B$ and $f^{-1} \circ f = i_A$.

- <u>f **is injective**</u>: Suppose $x, y \in A$ *s.t.* $f(x) = f(y)$.

  Apply f^{-1} to both sides, we get

$$\begin{gathered} f^{-1}(f(x)) = f^{-1}(f(y)) \\ (f^{-1} \circ f)(x) = (f^{-1} \circ f)(y) \\ (i_A)(x) = (i_A)(y) \\ x = y. \end{gathered}$$

- <u>f **is surjective**</u>: Let $b \in B$ [WTS: $\exists a \in A$ *s.t.* $f(a) = b$.]

  Choose $a = f^{-1}(b) \in A$.

  Taking f on both sides, we get

$$\begin{aligned} f(a) &= f(f^{-1}(b)) \\ &= (f \circ f^{-1})(b) \\ &= (i_B)(b) \\ &= b \end{aligned}$$

  So, $\exists a \in A$ *s.t.* $f(a) = b$. So, f is surjective. $\square$

($\Leftarrow$): Suppose $f \in \mathbf{F}(A, B)$ is bijective. [WTS: f is invertible.]

Let $b \in B$. Since f is surjective, $\exists a \in A$ *s.t.* $f(a) = b$.

Suppose $\exists c \in A$ *s.t.* $f(c) = b$.

Since f is injective, and $f(a) = f(c) = b$, we have $a = c$.

So, $\exists$ <u>unique</u> $a \in A$ *s.t.* $f(a) = b$.

Let's define $f^{-1}(b) = a$. Then, $f^{-1} \in \mathbf{F}(B, A)$. [WTS: f^{-1} is the inverse of f.]

- $f^{-1} \circ f = i_A$: Let $a \in A$. Suppose $f(a) = b$, then $f^{-1}(b) = a$.

$$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a.$$

  So, $f^{-1} \circ f = i_A$.

- $f \circ f^{-1} = i_B$: Let $b \in B$. Suppose $f^{-1}(b) = a$, then $f(a) = b$.

$$(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b.$$

  So, $f \circ f^{-1} = i_B$.

Therefore, f^{-1} is the inverse of f, or f is invertible.

■

Theorem 5.4.3. It turns out that a function can have at most one inverse function. Thus, we denote *the* inverse function of f by f^{-1}.

`Prove that a function can have at most one inverse function.`

***Proof* (7).**
Suppose $f \in \mathbf{F}(A, B)$
Suppose $g, h \in \mathbf{F}(B, A)$ *s.t.* $f \circ g = i_B, f \circ h = i_B, g \circ f = i_A$, and $h \circ f = i_A$.
Since $g \circ f = i_A$, compose by h on both sides, we get

$$(g \circ f) \circ h = i_A \circ h = h$$

and

$$(g \circ f) \circ h = g \circ (f \circ h) = g \circ i_B = g$$

So, we see that $g = h$.

■

`Consider the function` $f : \mathbb{R} \to \mathbb{R}$ `given by` $f(x) = x^3 + x - 2$. `Prove that` f `is invertible.`

***Proof* (8).**
We cannot compute the inverse function of f, but we will show that f is bijective.

- f **is injective**: Note that $f'(x) = 3x^2 + 1 > 0$. So, f is increasing on $\mathbb{R}$.
  Thus, if $f(x) = f(y)$, then $x = y$. □

- **f is surjective**: First note that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(x^3 + x - 2\right) = \infty$.

  Also, $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \left(x^3 + x - 2\right) = -\infty$.

  Let $y \in \mathbb{R}$ [WTS: $\exists x \in \mathbb{R}$ *s.t.* $f(x) = y$.]

  Since $\lim_{x\to\infty} f(x) = \infty$, eventually for large positive x, $f(x) > y$.

  So, $\exists b \in \mathbb{R}$ *s.t.* $f(b) > y$.

  Since $\lim_{x\to-\infty} f(x) = -\infty$, eventually for large negative x, $f(x) < y$.

  So, $\exists a \in \mathbb{R}$ *s.t.* $f(a) < y$.

  Since f is continuous on $[a, b]$ and $f(a) < y < f(b)$, $\exists x \in (a, b) \in \mathbb{R}$ *s.t.* $f(x) = y$.

  By the intermediate value theorem, f is surjective.

■

6 Infinite Sets

6.1 Countable Sets

Key Idea of this section: Use functions to compare sizes of infinite sets.

Theorem 6.1.1 (Pigeonhole Principle in the Language of Functions). Let A and B be finite sets and let $f : A \to B$ be any function.

1. If $|A| > |B|$, then f is not injective. (There are more pigeons than holes, i.e., $\exists p_1, p_2 \in A$ *s.t.* $f(p_1) = f(p_2)$ but $p_1 \neq p_2$.)

   Remark. f is injective $\implies |A| \leq |B|$

2. If $|A| < |B|$, then f is not surjective.(There are more holes and pigeons, so there are at least one empty hole).

   Remark. f is surjective $\implies |A| \geq |B|$

Corollary 6.1. Let A and B be finite sets, and $f : A \to B$ be a function. If $|A| \neq |B|$, then f is not a bijective. That is, f is bijective $\iff |A| = |B|$.

Remark. Whenever we have a bijection $f : A \to B$, we should consider both sets A and B to be of the same **size**. We can apply this to **infinite** sets A and B to compare their relative sizes: as long as we can find a bijection from A to B, we say A and B have the same *size*, even though A and B are infinite sets.

Prove that there exists a bijection between $\mathbb{Z}_{\geq 0}$, the set of non-negative integers, and $\mathbb{N}$, the set of natural numbers.

$$\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \cdots\}; \qquad \mathbb{N} = \{1, 2, 3, 4, \cdots\}$$

***Proof* (1).**
Let's define a function $f : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ as $f(n) = n - 1$.

- **Injective**: Let $a, b \in \mathbb{N}$ *s.t.* $f(a) = f(b)$.

  So, $a - 1 = b - 1$. That is, $a = b$.

- **Surjective**: Let $k \in \mathbb{Z}_{\geq 0}$.

  Choose $a = k + 1$. Note that since $k \geq 0,\ k + 1 \geq 1$. So, $k + 1 \in \mathbb{N}$.

  Also, $f(a) = f(k + 1) = (k + 1) - 1 = k$.

  So, f is surjective.

■

Remark. The inverse function of f is given by $g : \mathbb{Z}_{\geq 0} \to \mathbb{N}$ as $g(k) = k + 1$.

Definition 6.1.1 (Numerical Equivalence). Let A and B be sets in some universal set U. We say A and B are **numerically equivalent**, denoted $A \approx B$, if there exists a bijection $f : A \to B$. We also say A and B have the **same cardinality** ($|A| = |B|$).

Example 6.1.1. In Proof 6.1, we have shown that $\mathbb{N} \approx \mathbb{Z}_{\geq 0}$.

Let $\mathbb{Z}_{<0}$ be the set of negative integers. Prove that $\mathbb{N} \approx \mathbb{Z}_{<0}$.

$$\mathbb{Z}_{<0} : \{\cdots, -3, -2, -1\}; \qquad \mathbb{N} = \{1, 2, 3, \cdots\}$$

***Proof* (2).**
Define $f : \mathbb{N} \to \mathbb{Z}_{<0}$ as $f(n) = -n$.

- **Injective**: Let $a, b \in \S$ *s.t.* $f(a) = f(b)$. Then, $-a = -b$, so $a = b$. □

- **Surjective**: Let $k \in \mathbb{Z}_{<0}$. Then, $k \leq -1$, so $-k \geq 1$.

  Set $a = -k \in \mathbb{N}$.

  Then, $f(a) = f(-k) = -(-k) = k$. So, f is surjective.

■

Remark. The inverse function of f is given by $g : \mathbb{Z}_{<0} \to \mathbb{N}$ as $g(k) = -k$.

```
Let 𝔼 = 2ℤ be the set of even integers.  Prove that ℤ ≈ 𝔼.
```

$$\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}; \qquad \mathbb{E} = \{\cdots, -2, 0, 2, \cdots\}$$

***Proof* (3).**
Define $f : \mathbb{Z} \to \mathbb{E}$ as $f(n) = 2n$.

- **Injective**: Let $a, b \in \mathbb{Z}$ ***s.t.*** $f(a) = f(b)$. Then, $2a = 2b$, so $a = b$. □
- **Surjective**: Let $k \in \mathbb{E}$. Then, $\exists a \in \mathbb{Z}$ ***s.t.*** $k = 2a$.

  Note that $f(a) = 2a = k$. So, f is surjective.

■

Remark. The inverse function of f is defined as $g : \mathbb{E} \to \mathbb{Z}$ as $g(k) = \dfrac{k}{2}$.

Remark. There are as many integers as even numbers. They are infinite in the same way.

```
Prove that ℤ≥0 ≈ 𝔼.
```

$$\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \cdots\}; \qquad \mathbb{E} = \{\cdots, -6, -4, -2, 0, 2, 4, 6, 8, \cdots\}$$

***Proof* (4).**

Define $f : \mathbb{Z}_{\geq 0} \to \mathbb{E}$ as $f(n) = \begin{cases} n & \text{if } n \text{ is even} \\ -n-1 & \text{if } n \text{ is odd.} \end{cases}$

- **Injective**: Suppose $a, b \in \mathbb{Z}_{\geq 0}$ ***s.t.*** $f(a) = f(b)$.

  $\boxed{\text{Case 1}}$ Suppose a, b are both even, then $a = b$.

  $\boxed{\text{Case 2}}$ Suppose a, b are both odd, then $-a - 1 = -b - a$, so $a = b$.

  $\boxed{\text{Case 3}}$ Suppose a is even and b is odd. Then, $f(a) > 0$ and $f(b) < 0$.

  So, it is impossible for $f(a) = f(b)$. So, this case is impossible. □

- **Surjective**: Let $k \in \mathbb{E}$.

  $\boxed{\text{Case } 1}$ If $k \geq 0$. Then, $f(k) = k$ by definition of f.

  $\boxed{\text{Case } 2}$ If $k < 0$.

  Set $a = -k - 1$.

  Note that $k \leq -2$ (since k is a negative even integer), so $-k \geq 2$ and $-k - 1 \geq 1$.

  So, $a \in \mathbb{Z}_{\geq 0}$ and a is odd.

  So,

$$
\begin{aligned}
f(a) = -a - 1 = -(-k-1) - 1 \\
&= k + 1 - 1 \\
&= k
\end{aligned}
$$

  So, f is surjective.

■

Remark. The inverse of f is given by $g : \mathbb{E} \to \mathbb{Z}_{\geq 0}$ as $g(k) = \begin{cases} k & \text{if } k \geq 0 \\ -k - 1 & \text{if } k < 0. \end{cases}$

Lemma 6.1. Numerical equivalence defines an *equivalence relation* on the power set of U.

Proof **(5).**

- **Reflexive**: Let $A \subseteq U$ or $A \in \mathcal{P}(U)$

  Note that $i_A : A \to A$ is a bijection, so $A \approx A$. □.

- **Symmetric**: Let $A, B \subseteq U$.

  Suppose $A \approx B$. So, $\exists f : A \to B$ ***s.t.*** f is a bijection.

  We know that f is invertible and $f^{-1} \in \mathbf{F}(B, A)$.

  We shoed on a homework that f^{-1} is a bijection. So, $B \approx A$. □

- **Transitive**: Let $A, B, C \in U$.

  Suppose $A \approx B$ and $B \approx C$.

  So, $\exists f : A \to B$ and $g : B \to C$ ***s.t.*** f and g are bijections. Then, $g \circ f : A \to C$.

Showed if f and g are bijection, then $g \circ f$ is a bijection.

So, $A \approx C$.

■

`Use Lemma 6.1 to give a streamlined proof that` $\mathbb{N} \approx \mathbb{Z}$.

***Proof* (6).**
We've shown that $\mathbb{N} \approx \mathbb{Z}_{\geq 0}$ (Proof 6.1) and $\mathbb{Z}_{\geq 0} \approx \mathbb{E}$ (Proof 6.1).
So, by transitivity, we have $\mathbb{N} \approx \mathbb{E}$.
Since we also proved that $\mathbb{E} \approx \mathbb{Z}$ in Proof 6.1, again by transitivity, we have $\mathbb{N} \approx \mathbb{Z}$.

■

Remark. There are as many natural numbers as integers.

Notation 6.1. We write $|\mathbb{N}| = \aleph_0$ (aleph naught). This is also called the first infinity or the smallest infinity.

Definition 6.1.2 (Countably Infinite). A set A is said to be **countably infinite** if $A \approx \mathbb{N}$ (A has the same cardinality as $\mathbb{N}$).

Definition 6.1.3 (Countable Sets). It is said to be **countable** if A is countable infinite, or A is finite. If A is infinite, but $|\mathbb{N}| \neq |A|$, we say A is **uncountable**.

Remark. A set is countable infinite if and only if its elements can be arranged in a infinite list $\{a_1, a_2, a_3, \cdots\}$ (i.e., we can count them).

Example 6.1.2. As we have proven above, $\mathbb{Z}_{\geq 0}, \ \mathbb{Z}_{<0}, \ \mathbb{E}, \ \mathbb{Z}$ are all countable infinite. But $\mathbb{R}$ is uncountable.

Theorem 6.1.2. Let A be a countably infinite set and let B be an infinite subset of A. Then, B is countably infinite.

Corollary 6.2. Every subset of $\mathbb{Z}$ is countably infinite.

Theorem 6.1.3. Let A be an infinite set. Suppose there exists a surjection $f : \mathbb{N} \to A$. Then, A is countably infinite. That is, $|\mathbb{N}| \geq |A|$.

Theorem 6.1.4. Let A and B be countably infinite sets. Then $A \cup B$ is countably infinite.

***Proof* (7).**
Since A, B are countably infinite, $\exists f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$ that are bijective.
[WTS: $\exists h : \mathbb{N} \to A \cup B$ that is bijective]
Define h as follows:

$$h(1) = f(1), \quad h(2) = g(1), \quad h(3) = f(2), \quad h(4) = g(2), \cdots$$

Note that h may not be injective because it may be that $A \cap B \neq \varnothing$.
But we can show h is surjective.
Let $x \in A \cup B$. Then by definition, $x \in A$ or $x \in B$.
WLOG, suppose $x \in A$. Since f is surjective, $\exists k \in \mathbb{N}$ ***s.t.*** $f(k) = a$.
But then, $\exists m \in \mathbb{N}$ ***s.t.*** $h(m) = f(k) = a$. So h is surjective.

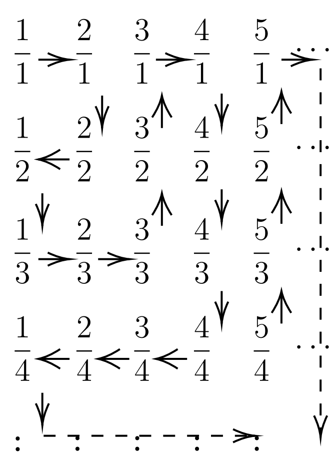
■

Extension. Let $A_1, A_2, \cdots, A_k$ be a finite collection of countably infinite sets. Then $A_1 \cup A_2 \cup \cdots \cup A_k$ is a countably infinite set.

```
Show that ℚ is countably infinite.
```

***Proof* (8).**

We will show that $\mathbb{Q}^+$ (the set containing positive rational numbers) is countably infinite. We can list elements of $\mathbb{Q}^+$ like so: $\left\{ \frac{p}{q} \mid p, q \in \mathbb{N} \right\}$



That is, define $f : \mathbb{N} \to \mathbb{Q}^+$ as follows:

$$f(1) = \frac{1}{1},\ f(2) = \frac{2}{1},\ f(3) = \frac{2}{2},$$

$$f(4) = \frac{1}{2},\ f(5) = \frac{1}{3},\ f(6) = \frac{2}{3},\ \cdots$$

Again, this function is not injective because each integer shows up in the outputs infinitely many times.

However, f is surjective. So, $\mathbb{Q}^+$ is countably infinite.
Note that since $\exists$ bijection between $\mathbb{Q}^+$ and $\mathbb{Q}^-$, we see that Q^- is countably infinite.
So, $\mathbb{Q}^+ \cup \mathbb{Q}^-$ is countably infinite.
Finally, $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$ is countably infinite.

■

Theorem 6.1.5. If A and B are countably infinite, then so is $A \times B$.

***Proof* (9).**
Since A and B are countably infinite, $\exists$ bijections $g : \mathbb{N} \to A$ and $h : \mathbb{N} \to B$.
$\forall i \in \mathbb{N}$, we call $g(i) = a_i \in A$ and $h(i) = b_i \in B$.
Note that $A \times B = \{(a_i, b_j) \mid i, j \in \mathbb{N})\}$. So we can list them like so:

$$\begin{tikzcd}
(a_1, b_1) \arrow[d] & (a_1, b_2) \arrow[r] & (a_1, b_2) \arrow[d] & (a_1, b_2) & \cdots \arrow[dddd, dashed] \\
(a_2, b_1) \arrow[r] & (a_2, b_2) \arrow[u] & (a_2, b_2) \arrow[d] & (a_2, b_2) \arrow[u] & \cdots \\
(a_3, b_1) \arrow[d] & (a_3, b_2) \arrow[l] & (a_3, b_2) \arrow[l] & (a_3, b_2) \arrow[u] & \cdots \\
(a_4, b_1) \arrow[r] & (a_4, b_2) \arrow[r] & (a_4, b_2) \arrow[r] & (a_4, b_2) \arrow[u] & \cdots \\
\vdots & \vdots & \vdots & \vdots & {} \arrow[llll, dashed]
\end{tikzcd}$$

That is, define $f : \mathbb{N} \to A \times B$ as follows:

$$f(1) = (a_1, b_1), \quad f(2) = (a_2, b_1) \quad f(3) = (a_2, b_2), \quad f(4) = (a_1, b_2), \cdots$$

Since f is surjective, $A \times B$ is countably infinite.

■

Extension. The cartesian product of any finite number of countably infinite sets is countably infinite. That is, $A_1 \times A_2 \times A_3 \cdots \times A_k$ is countably infinite, where $A_1, A_2, A_3, c \ldots, A_k$ are countably infinite.

***Proof* (10).**
The proof of this Extension is left as an exercise.
We can prove this using Theorem 6.1.5 and mathematical induction.

■

6.2 Uncountable Sets

Theorem 6.2.1 (Cantor's Diagonalization Arguments). There is no bijection from $\mathbb{N} \to \mathbb{R}$.

***Proof* (1).**
The very basic idea: Proceed by contradiction: Assume $\exists f : \mathbb{N} \to \mathbb{R}$ that is a bijection.
Show that $\exists a \in \mathbb{R}$ ***s.t.*** $\forall n \in \mathbb{N},\ f(n) \neq a$.
So that contradicts with the surjectivity of f.

■

Remark. $|\mathbb{N}| \neq |\mathbb{R}|$. These are two different infinities.

Example 6.2.1. Show that the following pairs of sets have the same cardinalities by finding a bijection between them.

1. $A = (0, 1)$ and $B = (0, 2)$

   Answer. Define $f : A \to B$ by $f(x) = 2x$. □

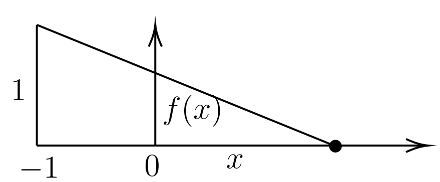
2. $A = (0, \infty)$ and $B = (0, 1)$

   Answer. Define $f : A \to B$ that is bijective as:

   $\boxed{\text{Method 1}}$ $f(x) = \dfrac{2}{\pi} \arctan(x)$.

   $\boxed{\text{Method 2}}$ $f(x) = \dfrac{1}{3^x}$.

   $\boxed{\text{Method 3}}$ Using similar triangles



$$\frac{f(x)}{1} = \frac{x}{x+1} \implies f(x) = \frac{x}{x+1}$$

□

Remark. To show $(a, b) \approx (c, d)$, define $f(x)$ as the straight line between (a, b) and (c, d).

Definition 6.2.1 (Comparing Cardinalities). Let A and B be sets. We say

1. $|A| = |B|$ if there exists a bijection from A to B.

2. $|A| \leq |B|$ if there exists an injection from A to B.

3. $|A| < |B|$ if there exists an injection, but no surjection from A to B.

`Show` $|\mathbb{Q}| = |\mathbb{N}| < |\mathbb{R}|$

***Proof* (2).**
We showed $\mathbb{Q}$ is countably infinite. So, $\mathbb{Q} \approx \mathbb{N}$, or $|\mathbb{Q}| = |\mathbb{N}|$.
(*To show* $|\mathbb{N}| < |\mathbb{R}|$, *we need to show that* $\exists$ *injection from* $\mathbb{N} \to \mathbb{R}$ *but now surjection.*)
Define $f : \mathbb{N} \to \mathbb{R}$ as $f(n) = n$. Then, $f \in \mathbf{F}(\mathbb{N}, \mathbb{R})$ and f is clearly injective.
But by Cantor's Argument, $\nexists$ surjection from $\mathbb{N} \to \mathbb{R}$ (Theorem 6.2.1), we have $|\mathbb{N}| < |\mathbb{R}|$.
$\blacksquare$

Remark. Here we show that the infinity of $\mathbb{R}$ is bigger than that of $\mathbb{N}$.

Theorem 6.2.2. For any set A, $|A| < |\mathcal{P}(A)|$.

***Proof* (3).**
$\boxed{\text{Case 1}}$ If A is finite, then $|A| = n$ and $|\mathcal{P}(A)| = 2^n$.
Since $n < 2^n$ (*can be proven using mathematical induction*), so $|A| < |\mathcal{P}(A)|$.
$\boxed{\text{Case 2}}$ Suppose A is infinite.
WTS: $\exists$ *injection* $f : A \to \mathcal{P}(A)$ *AND* $\nexists$ *surjection* $f : A \to \mathcal{P}(A)$.

- **Injection exists**: Let's define $f : A \to \mathcal{P}(A)$ as $f(a) = \{a\}$.

  Then, clearly if $a, b \in A$ *s.t.* $f(a) = f(b)$, then $\{a\} = \{b\}$

  So, $a = b$. $\square$

- **No surjection exists**:

  Suppose for the sake of contradiction that $\exists f : A \to \mathcal{P}(A)$ that is surjective.

  Note that $\forall a \in A,\ f(a) \in \mathcal{P}(A)$.

  That is, $\forall a \in A,\ f(a) \subseteq A$.

  So, $\forall a \in A$, either $a \in f(a)$ or $a \notin f(a)$.

  Define $B = \{a \in A \mid a \notin f(a)\} \subseteq A$. So, $B \in \mathcal{P}(A)$.

  Since f is surjective and $B \in \mathcal{P}(A)$, $\exists a_0 \in A$ *s.t.* $f(a_0) = B$.

  $\boxed{\text{Case 1}}$ Suppose $a_0 \in B$. Then, by definition, $a_0 \notin f(a_0) = B$.

$$\divideontimes \text{ This contradicts that } a_0 \in B$$

$\boxed{\text{Case 2}}$ Suppose $a_0 \notin B$. Then, by definition, $a_0 \in f(a_0) = B$.

$$\divideontimes \text{ This contradicts that } a_0 \notin B.$$

So, it must be that $\forall a \in A,\ f(a) \neq B$.

So, f cannot be surjective.

■

Remark. From Theorem 6.2.2, we have the following inequality that contains infinitely many infinities:

$$\underbrace{|\mathbb{N}|}_{\text{countable}} < \overbrace{|\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| < \cdots}^{\text{uncountable}}$$

`Show that there are infinitely many irrational numbers.`

***Proof* (4).**

We know that $\sqrt{2}$ is irrational.

Consider $f : \mathbb{N} \to \mathbb{Q}'$, $\mathbb{Q}'$ the set of irrational numbers, *s.t.* $f(n) = \sqrt{2} + n$.

Suppose for the sake of contradiction that $\sqrt{2} + n = q$, where $q \in \mathbb{Q}$.

Then, $\sqrt{2} = q - n$.

Since $q \in \mathbb{Q}$ and $n \in \mathbb{N} \subseteq \mathbb{Q}$, so we know $q - n \in \mathbb{Q}$.

$$\divideontimes \text{ This contradicts with the fact that } \sqrt{2} \in \mathbb{Q}'.$$

Note that f is injective. Suppose $n, m \in \mathbb{N}$ ***s.t.*** $f(n) = f(m)$.

Then, $\sqrt{2} + n = \sqrt{2} + m$. So, $m = n$.

So, $|\mathbb{N}| \leq |\mathbb{Q}'|$. So, $\mathbb{Q}'$ is an infinite set.

■

`Show that the set of irrational numbers,` $\mathbb{Q}'$`, is uncountable.`

***Proof* (5).**

Recall that $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$.

Suppose for the sake of contradiction, $\mathbb{Q}'$ is countable.

Then, $\mathbb{R}$ is the union of two countable sets, so $\mathbb{R}$ is countable,

⋇ We know $\mathbb{R}$ is uncountable.

So, it must be that $\mathbb{Q}'$ is uncountable.

■

Remark.

$$|\mathbb{R}| = |\mathbb{Q}'| = \underbrace{|(a,b)|}_{\text{open interval}}.$$