1 Statements

1.1 Class Handout, Chapter 1.3, Implications.

`Let` a, b, `and` c `be integers, with` a `and` b `non-zero. If` $(ab) \mid (ac)$`, then` $b \mid c$.

Proof 1.

Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$ and $b \neq 0$. Suppose $(ab) \mid (ac)$. Then $\exists k \in \mathbb{Z}$ *s.t.* $ac = (ab)k$. Divide both sides of the equation by a:

$$c = bk.$$

Since $k \in \mathbb{Z}$, by definition of divides, $b \mid c$.

■

1.2 Class Handout, Chapter 1.4, Contrapositive and Converse

`Prove that for all real numbers` a `and` b`, if` $a \in \mathbb{Q}$ `and` $ab \notin \mathbb{Q}$`, then` $n \notin Q$.

Proof 2.

Let $a, b \in \mathbb{Q}$. Assume for the sake of contradiction that if $a \in \mathbb{Q}$ and $ab \notin \mathbb{Q}$, we have $b \in \mathbb{Q}$. Then, $\exists p, q, m, n \in \mathbb{Z}$ *s.t.* $a = \frac{m}{n}$ and $b = \frac{p}{q}$. Hence,

$$ab = \frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq}$$

As $mp, nq \in \mathbb{Z}$, $ab \in \mathbb{Q}$.

$\divideontimes$ This contradicts with the fact that $ab \notin \mathbb{Q}$.

So, b must not be rational.

■

1.3 Chapter 1.1 # 7(c)

`Prove the square of an even integer is divisible by 4.`

Proof 3.

Suppose $x \in \mathbb{Z}$ is even. Then $\exists k \in \mathbb{Z}$ *s.t.* $x = 2k$. Then, $x^2 = (2k)^2 = 4k^2$. Since $k^2 \in \mathbb{Z}$, we have $4 \mid 4k^2$.

■

Theorem 1.1 (Archimedean Principle) *For every real number x, there is an integer n, such that $n > x$.*

1.4 Chapter 1.1 # 11

`For every positive real number` ε`, there exists a positive integer` N `such that` $\frac{1}{n} < \varepsilon$ `for all` $n \geq N$.

Proof 4.

Suppose $\varepsilon \in \mathbb{R}$ and $\varepsilon > 0$. Since $\varepsilon \in \mathbb{R}$, we have $\frac{1}{n} \in \mathbb{R}$. Then, by Archimedean Principle, $\exists n \in \mathbb{Z}$ *s.t.* $n > \frac{1}{\varepsilon}$. Hence, $n\varepsilon > 1$ or $\varepsilon > \frac{1}{n}$.

Suppose $N \in \mathbb{Z}$ *s.t.* $N = \left\lceil \frac{1}{\varepsilon} \right\rceil$, *where* $\left\lceil \frac{1}{\varepsilon} \right\rceil$ *means the integer greater to* $\frac{1}{\varepsilon}$ *if* $\frac{1}{\varepsilon} \notin \mathbb{Z}$, *and the integer equals to* $\frac{1}{\varepsilon}$ *if* $\frac{1}{\varepsilon} \in \mathbb{Z}$. Hence, $N \geq \frac{1}{\varepsilon}$. As $n > \frac{1}{\varepsilon}$, we have $n \geq N$

$\blacksquare$

1.5 Chapter 1.1 # 12

`Use the Archimedean Principle (Theorem 1.1) to prove if` x `is a real number, then there exists a positive integer` n `such that` $-n < x < n$.

Proof 5.

Suppose $x \in \mathbb{R}$.

$\boxed{\text{Case 1}}$ If $x > 0$, then $-x < 0$ (i.e., $-x < 0 < x$). By the Archimedean Principle, $\exists n \in \mathbb{Z}$ *s.t.* $n > x$. Multiply (-1) on both sides of the inequality:

$$-n < -x$$

As $-x < 0 < x$,

$$-n < -x < 0 < x < n,$$

which means $-n < x < n$, and n is positive.

$\boxed{\text{Case 2}}$ If $x < 0$, then $-x > 0$ (i.e., $-x > 0 > x$) Since $x \in \mathbb{R}$, we have $-x \in \mathbb{R}$. By the Archimedean Principle, $\exists n \in \mathbb{Z}$ *s.t.* $n > -x$. Multiply (-1) on both sides of the inequality:

$$-n < x$$

As $x < 0 < -x$,

$$-n < x < 0 < -x < n,$$

which means $-n < x < n$, and n is positive. In all cases, we have proven that $x \in \mathbb{R} \implies \exists n \in \mathbb{Z},\ n > 0$ *s.t.* $-n < x < n$.

■

1.6 Chapter 1.1 # 13

`Prove that if` x `is a positive real number, then there exists a positive integer` n `such that` $\frac{1}{n} < x < n$.

Proof 6.

Suppose $x \in \mathbb{R},\ x > 0$

$\boxed{\text{Case 1}}$ If $0 < x \leq 1$, then $\frac{1}{x} \geq 1$. Hence, $x \leq 1 \leq \frac{1}{x}$. As $x \in \mathbb{R}$, $\frac{1}{x} \in \mathbb{R}$, then by the Archimedean Principle (Theorem 1.1):

$$\exists n \in \mathbb{Z} \textit{ s.t. } n > \frac{1}{x}.$$

Hence, $nx > 1$ or $x > \frac{1}{n}$. As $x \leq \frac{1}{x}$, $n > \frac{1}{x}$, and $x > \frac{1}{n}$, we have

$$\frac{1}{n} < x < n.$$

$\boxed{\text{Case 2}}$ If $x > 1$, then $0 < \frac{1}{x} < 1$. Hence, $\frac{1}{x} < 1 < x$. As $x \in \mathbb{R}$, by the Archimedean Principle:

$$\exists n \in \mathbb{Z} \textit{ s.t. } n > x > 0$$

Hence, $\frac{1}{n} < \frac{1}{x}$. As $\frac{1}{x} < x$, $\frac{1}{n} < \frac{1}{x}$, and $n > x$, we have

$$\frac{1}{n} < x < n$$

In all cases, we proven that $x \in \mathbb{R},\ x > 0 \implies \exists n \in \mathbb{Z}, n > 0$ *s.t.* $\frac{1}{n} < x < n$.

■

1.7 Handout Chapter 1.4-2 More Contradictions and Equivelance

`There are no positive integer solutions to the equation` $x^2 - y^2 = 10$.

Proof 7.

Assume for the sake of contradiction that there are positive integer solutions to the equation $x^2 - y^2 = 10$. Suppose $\exists x, y \in \mathbb{Z}$ and $x > 0,\ y > 0$ *s.t.* $x^2 - y^2 = 10$. Then, we have $x^2 = 10 + y^2$. Since

$x > 0,\ x^2 > 0$, we have $10 + y^2 > 0$. Then, $y^2 > -10$.

$\divideontimes$ This contradicts with the fact that $y^2 \geq 0$ if $y \in \mathbb{Z}$.

So, our assumption is wrong. There must be no positive integer solutions to the equation $x^2 - y^2 = 10$.

■

1.8 Handout Chapter 1.4-2 More Contradictions and Equivelance

`Show that if` $a \in \mathbb{Q}$ `and` $b \in \mathbb{Q}'$`, then` $a + b \in \mathbb{Q}'$

Remark *The notation $\mathbb{Q}$ means the set for rational numbers, and $\mathbb{Q}'$ means the set for irrational numbers.*

Proof 8.

Suppose $a \in \mathbb{Q}$ and $b \in \mathbb{Q}'$ Assume for the sake of contradiction that $a + b \in \mathbb{Q}$. Then, $\exists m, n, p, q \in \mathbb{Z}$ such that $a = \dfrac{m}{n}$ and $a + b = \dfrac{p}{q}$. Then,

$$b = \frac{p}{q} - a = \frac{p}{q} - \frac{m}{n} = \frac{pn - mq}{qn} \in \mathbb{Q}$$

Since $pn - mq \in \mathbb{Q}$ and $qn \in \mathbb{Z}$, we have $b = \dfrac{pn - mq}{qn} \in \mathbb{Q}$.

$\divideontimes$ This contradicts with the fact that $b \in \mathbb{Q}'$.

So, $a + b$ must be irrational.

■

1.9 Handout Chapter 1.4-2 More Contradictions and Equivalence

`If` $n \in \mathbb{N}$ `and` $2^n - 1$ `is prime, then` n `is prime.`

Proof 9.

We will prove the contrapositive: if n is not prime, then $2^n - 1$ is not prime. Suppose n is not prime. Then, $\exists a, b \in \mathbb{Z}$ with $1 < a, b < n$ *s.t.* $n = ab$. Then, $2^n - 1 = 2^{ab} = (2^a)^b - 1$. Notice that for $x^w - 1$, by polynomial long division, have

$$x^w - 1 = (x - 1)\left(x^{w-1} + x^{w-2} + \cdots + 1\right),$$

Substitute $x = 2^a$ and $w = b$, we have

$$2^n - 1 = (2^a - 1)\left[(2^a)^{b-1} + (2^a)^{b-2} + \cdots + 1\right].$$

Since $(2^a - 1) \in \mathbb{Z}$ and $\left[(2^a)^{b-1} + (2^a)^{b-2} + \cdots + 1\right] \in \mathbb{Z}$, we see that $2^n - 1$ is not prime.

■

1.10 Exam 1 Review 1-b-i

`Prove that` $[P \wedge (P \implies Q)] \implies Q$.

Proof 10.

P	Q	$P \Rightarrow Q$	$P \wedge (P \Rightarrow Q)$	$[P \wedge (P \Rightarrow Q)] \implies P$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

■

1.11 Exam 1 Review 1-b-ii

`Prove that` $[Q \wedge (P \implies Q)] \implies P$.

Proof 11.

P	Q	$P \Rightarrow Q$	$Q \wedge (P \Rightarrow Q)$	$[Q \wedge (P \Rightarrow Q)] \implies Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	F	T

■

1.12 Exam 1 Review 2-a

`Given statements` P `and` Q`, prove` $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$.

Proof 12.

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

■

1.13 Exam 1 Review 2-b

`There is no smallest integer.`

Proof 13.

Assume for the sake of contradiction that there exists a smallest integer n. Hence, $\forall x \in \mathbb{Z}$, we have $x \geq n$. Notice that if $n > 0$, we have $0 \in \mathbb{Z}$ and $0 < n$. Hence, $n = 0$ cannot be the smallest integer ($\divideontimes$) Therefore, n most be smaller than 0. Suppose $m = -n$. Since $n \in \mathbb{Z}$, $m = -n \in \mathbb{Z} \in \mathbb{R}$ By the Archimedean Principle (Theorem 1.1), $\exists k \in \mathbb{Z}$ *s.t.* $k > m$. Hence, $k > -n$. Multiply (-1) on both sides of the inequality:

$$-k < n.$$

As $k \in \mathbb{Z}$, $-k \in \mathbb{Z}$. Then $\exists -k \in \mathbb{Z}$ *s.t.* $-k < n$.

$\divideontimes$ This contradicts with our assumption that n is the smallest integer.

Hence, our assumption must be wrong. There is no smallest integer.

■

1.14 Exam 1 Review 2-c

`The number` $\log_2 3$ `is irrational.`

Proof 14.

Assume for the sake of contradiction that $\log_2 3$ is irrational. By definition, $\exists p, qin\mathbb{Z}$, with $q \neq 0$ *s.t.* $\log_2 3 = \frac{p}{q}$. Observe that $\log_2 3 \neq 0$. Then $p \neq 0$ as well. By definition of logarithm,

$$2^{p/q} = 3$$
$$(2^p)^{1/q} = 3$$

Raise two sides of the equation to the power of q:

$$2^p = 3^q$$

As $p \neq 0$ and $q \neq 0$, 2^p and 3^q are not 1 $\forall p, q \in \mathbb{Z}$. Hence, 2^p is even $\forall p \in \mathbb{Z}$ and 3^q is odd $\forall q \in \mathbb{Z}$.

⋇ This contradicts with the fact that an even number cannot equal to an odd number.

Hence, our assumption is wront. The number $\log_2 3$, then, must be irrational.

■

> *1.15 Exam 1 Review 2-d*
>
> `There is a rational number` a `and an irrational number` b `such that` a^b `is rational.`

Proof 15.

Observe that 1 is a rational number and π is an irrational number. Suppose $a = 1$ and $b = \pi$, we have $a^b = a^\pi = 1$, which is irrational.

■

Proof 16.

Recall that we have proven in the previous proof, we have proven that $\log_2 3$ is an irrational number. Recall the definition of logarithm and exponents, we have

$$2^{\log_2 3} = 3$$

Hence, we find a pair of a and b that satisfies the requirement.

■

> *1.16 Exam 1 Review 2-e*
>
> `For all integers` n`, the number` $n + n^2 + n^3 + n^4$ `is even.`

Proof 17.

Suppose $n \in \mathbb{Z}$.

$\boxed{\text{Case 1}}$ If n is even.Suppose $n = 2k$ *f.s.* $k \in \mathbb{Z}$. Then,

$$\begin{aligned} n + n^2 + n^3 + n^4 &= (2k) + (2k)^2 + (2k)^3 + (2k)^4 \\ &= 2k + 4k^2 + 8k^3 + 16k^4 \\ &= 2(k + 2k^2 + 4k^3 + 8k^4) \end{aligned}$$

Since $(k + 2k^2 + 4k^3 + 8k^4) \in \mathbb{Z}$, we have $2(k + 2k^2 + 4k^3 + 8k^4)$ is even. Hence, $n + n^2 + n^3 + n^4$ is even when n is even.

$\boxed{\text{Case 2}}$ If n is odd. Suppose $n = 2k + 1$ *f.s.* $k \in \mathbb{Z}$. Then,

$$\begin{aligned} n + n^2 + n^3 + n^4 &= (2k + 1) + (2k + 1)^2 + (2k + 1)^3 + (2k + 1)^4 \\ &= 2k + 1 + 4k^2 + 4k + 1 + 8k^3 + 12k^2 + 6k + 1 + 16k^4 + 32k^3 + 24k^2 + 8k + 1 \\ &= 16k^4 + 40k^3 + 40k^2 + 20k + 4 \\ &= 2(8k^4 + 20k^3 + 20k^2 + 10k + 2) \end{aligned}$$

Since $(8k^4 + 20k^3 + 20k^2 + 10k + 2) \in \mathbb{Z}$, we have $2(8k^4 + 20k^3 + 20k^2 + 10k + 2)$ is even. Hence, $n + n^2 + n^3 + n^4$ is even when n is odd.

Since integers can either be even or odd, and we have proven $n + n^2 + n^3 + n^4$ is even in either case, $n + n^2 + n^3 + n^4$ is even for all integers.

■

Definition 1.1 (Perfect Square) *A perfect square is an integer n for which there exists an integer m such that $n = m^2$.*

1.17 Exam 1 Review 2-f

`If` n `is a positive integer such that` n `is in the form` $4k+2$ `or` $4k+3$`, then` n `is not a perfect square.`

Proof 18.

We will prove the contrapositive of the statement: "If n is a perfect square, then n is a positive integer of the form $4k$ or $4k + 1$ *f.s.* $k \in \mathbb{Z}$." Suppose n to be a perfect square, then $\exists m \in \mathbb{Z}$ *s.t.* $n = m^2$.

$\boxed{\text{Case 1}}$ Suppose m is even, then $m = 2t$ *f.s.* $t \in \mathbb{Z}$.

$$n = m^2 = (2t)^2 = 4t^2 > 0.$$

Let $k = t^2$. Since $t^2 \in \mathbb{Z}$, we have $k \in \mathbb{Z}$.Hence, n is positive and is in the form of $4k$.

$\boxed{\text{Case 2}}$ Suppose m is odd, then $m = 2t + 1$ *f.s.* $t \in \mathbb{Z}$.

$$n = m^2 = (2t + 1)^2 = 4t^2 + 4t + 1 = 4(t^2 + t) + 1 > 1$$

Let $k = t^2 + t$. Since $(t^2 + t) \in \mathbb{Z}$, we have $k \in \mathbb{Z}$. Hence, n is in the form of $4k + 1$. Hence, we prove the contrapositive of the original statement to be true, which means our original statement is also true.

■

1.18 Exam 1 Review 2-g

`For any integer` n, $3 \mid n$ `if and only if` $3 \mid n^2$.

Proof 19.

Suppose $n \in \mathbb{Z}$.

($\Rightarrow$) Suppose $3 \mid n$. Then, $\exists k \in \mathbb{Z}$ *s.t.* $n = 3k$. Then, $n^2 = (3k)^2 = 9k^2 = 3(3k^2)$. Since $3k^2 \in \mathbb{Z}$, by definition, $3 \mid n^2$. □

($\Leftarrow$) WTS: $3 \mid n^2 \implies 3 \mid n$. We will prove the contrapositive: If $3 \nmid n$, then $3 \nmid n^2$ Suppose $3 \nmid n$.

$\boxed{\text{Case 1}}$ Suppose $n = 3m+1$ *f.s.* $m \in \mathbb{Z}$. Then, $n^2 = (3m+1)^2 = 9m^2+6m+1$ Since $9m^2+6m+1$ cannot be written in the form of $3k$ *f.s.* $k \in \mathbb{Z}$, by definition, $3 \nmid n^2$.

$\boxed{\text{Case 2}}$ Suppose $n = 3m + 2$ *f.s.* $m \in \mathbb{Z}$Then, $n^2 = (3m + 2)^2 = 9m^2 + 12m + 4$Since $9m^2 + 12m + 4$ cannot be written in the form of $3k$ for some $k \in \mathbb{Z}$, by definition, $3 \nmid n^2$. Hence, we proved the contrapositive, and thus the original statement is true.

Therefore, $n \mid n \iff 3 \mid n^2$.

■

1.19 Exam 1 Review 2-h

`There exists an integer` n `such that` $12 \mid n^2$ `but` $12 \nmid n$.

Proof 20.

Observe that if we take $n = 6$, we have $n^2 = 36$. Since $n^2 = 36 = 3 \times 12$, we know $12 \mid n^2$. However,

$12 \nmid 6$ since 6 cannot be written as $12k$ for all $k \in \mathbb{Z}$. Hence, there exists an integer $n = 6$ *s.t.* $12 \mid n^2$ but $12 \nmid n$.

■

1.20 Exam 1 Review 2-i
`For every integer` a`, the numbers` a `and` $(a+1)(a-1)$ `have opposite parity.`

Proof 21.

Suppose $a \in \mathbb{Z}$.

$\boxed{\text{Case 1}}$ Suppose a is even. Then $a = 2k$ *f.s.* $k \in \mathbb{Z}$. Then,

$$(a+1)(a-1) = a^2 - 1 = (2k)^2 - 1 = 4k^2 - 1 = 2(2k^2) - 1.$$

Since $2k^2 \in \mathbb{Z}$, we have $(a+1)(a-1)$ is odd. That is, a and $(a+1)(a-1)$ have opposite parity.

$\boxed{\text{Case 2}}$ Suppose a is odd. Then $a = 2k+1$ *f.s.* $k \in \mathbb{Z}$. Hence,

$$(a+1)(a-1) = a^2 - 1 = (2k+1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 2(2k^2 + 2k).$$

Since $2k^2 + 2k \in \mathbb{Z}$, we have $(a+1)(a-1)$ is even. As a result, a and $(a+1)(a-1)$ have opposite parity.

In both cases, we've shown that a and $(a+1)(a-1)$ have opposite parity.

■

1.21 Exam 1 Review 2-j
`Suppose` $x \in \mathbb{R}$`. If` x^2 `is irrational, then` x `is irrational.`

Proof 22.

We will prove the contrapositive: "If x is rational, then x^2 is rational." Suppose $x \in \mathbb{Q}$, then $x = \dfrac{p}{q}$ *f.s.* $p, q \in \mathbb{Z}$, assuming p and 1 have no common factors and $q \neq 0$. Then,

$$x^2 = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2}.$$

As $p, q \in \mathbb{Z}$, we have $p^2, q^2 \in \mathbb{Z}$. Hence, $x^2 = \dfrac{p^2}{q^2} \in \mathbb{Q}$. Therefore, if x is rational, so is x^2.

■

1.22 Exam 1 Review 2-k

`For any integers` a `and` b`, if` ab `is even, then` a `is even or` b `is even.`

Proof 23.

We will prove the contrapositive: "If a is odd and b is odd, then ab is odd." Suppose $a, b \in \mathbb{Z}$ and a and b are both odd. Then, $\exists k, l \in \mathbb{Z}$ *s.t.* $a = 2k + 1 \quad \text{and} \quad b = 2l + 1$. Then,

$$ab = (2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1.$$

Since $2kl + k + l \in \mathbb{Z}$, we have ab is odd.

■

1.23 Exam 1 Review 2-l

`For` $n \in \mathbb{N}$`,` n`,` $n+2$`, and` $n+4$ `are all prime if and only if` $n = 3$`.`

Proof 24.

($\Rightarrow$) WTS: $n, n+2$, and $n+4$ are all prime $\implies n = 3$. We will prove the contrapositive: $n \neq 3 \implies n, n+2$, or $n+4$ is not prime.

$\boxed{\text{Case 1}}$ Suppose $0 < n < 3$.

① If $n = 1$, then $n = 1$ is not a prime.

② If $n = 2$, then $n = 2$ is a prime number, but $n + 2 = 2 + 2 = 4$ is not a prime.

Hence, if $0 < n < 3$, $n, n+2$, or $n+4$ is not a prime.

$\boxed{\text{Case 2}}$ Suppose $n > 3$.

① If $n = 3k$ *f.s.* $k \in \mathbb{Z}$, then n is not a prime because $3 \mid n$.

② If $n = 3k + 1$ *f.s.* $k \in \mathbb{Z}$, then $n + 2 = 3k + 1 + 2 = 3k + 3 = 3(k + 1)$. Since $k + 1 \in \mathbb{Z}$, we have $3 \mid n + 2$. Then, $n + 2$ is not a prime.

③ If $n = 3k + 2$ *f.s.* $k \in \mathbb{Z}$, then $n + 4 = 3k + 2 + 4 = 3k + 6 = 3(k + 2)$. Since $k + 2 \in \mathbb{Z}$, we know that $3 \mid n + 4$. Therefore, $n + 4$ is not a prime.

Hence, if $n > 3$, we also have $n, n+2$, or $n+4$ is not a prime.

In both cases, we have proven that if $n \neq 3$, then $n, n+2$, or $n+4$ is not a prime. □

($\Leftarrow$) Note that when $n = 3$, we have $n + 2 = 3 + 2 = 5$ and $n + 4 = 3 + 4 = 7$. Since $3, 5$, and 7 are all primes, we have shown that when $n = 3$, n, $n + 2$, and $n + 4$ are all primes.

■

1.24 Exam 1 Review 3-a

`Prove or disprove: Every real number is less than or equal to its square.`

Disproof 25.

We will prove the negation: "Some real number is greater than its square." Observe that when $x = 0.1$, then $x^2 = (0.1)^2 = 0.01$. Since $0.01 < 0.1$, we have $x = 0.1 \in \mathbb{R}$ is greater than its square. Since the negation is true, the original statement is then false.

■

1.25 Exam 1 Review 3-b

`Prove or disprove: The sum of two integers is never equal to their product.`

Disproof 26.

We will prove the negation: "The sum of some integers is equal to their product." Suppose $p, q \in \mathbb{Z}$, and their sum equals to their product. Then, $p + q = pq$. Divide p on both sides: $q = 1 + \frac{q}{p}$. Observe that when $p = 2$, we have $q = 1 + \frac{q}{2}$. So, $2q = 1 + q$, or $q = 2$. Hence, $p + q = 2 + 2 = 4$ and $pq = 2 \times 2 = 4$. Therefore, we've found integers $p = 2$ and $q = 2$ such that $p + q = pq$.

■

1.26 Exam 1 Review 3-c

`Prove or disprove: There exists a non-zero integer whose cube equals its negative.`

Disproof 27.

We will prove the negation: "For all non-zero integers, their cubes do not equal their negations." Assume for the sake of contradiction that there exists a non-zero integer whose cube equals its negative. Suppose $x \in \mathbb{Z}$ and $x \neq 0$ *s.t.* $x^3 = -x$. So we have $x^3 + x = 0$, or $x(x^2 + 1) = 0$. Then, $x = 0$ or $x^2 + 1 = 0$. As $x \neq 0$, it must be that $x^2 + 1 = 0$, or $x^2 = -1$.

$$\divideontimes \text{ This contradicts with the fact that } \forall x \in \mathbb{Z}, x^2 \geq 0 > -1.$$

So, our assumption is incorrect. For all non-zero integers, their cubes do not equal their negatives.

■

1.27 Exam 1 Review 3-d

`Prove or disprove: Fall all` $x \in \mathbb{R}$, $x \leq x^2$ `or` $0 \leq x < 1$.

Proof 28.

Suppose $x \in \mathbb{R}$.

$\boxed{\text{Case 1}}$ Suppose $0 \leq x < 1$. Then, x satisfies the requirement.

$\boxed{\text{Case 2}}$ Suppose $x < 0$, then $x^2 > 0$. Therefore, $x < 0 < x^2$.

$\boxed{\text{Case 3}}$ Suppose $x \geq 1$. Multiply the inequality by x on both sides, we have: $x \cdot x \geq x$ or $x^2 \geq x$. Hence, $x \leq x^2$.

In all cases, we've proven that $\forall x \in \mathbb{R}, x \leq x^2$ or $0 \leq x < 1$.

■

1.28 Chapter 1.4 # 20-a

`Let` n `be an integer. Prove that` n `is even if and only if` n^3 `is even.`

Proof 29.

($\Rightarrow$) WTS: n is even $\implies$ n^3 is even. Suppose n is even. Then $n = 2k$ *f.s.* $k \in \mathbb{Z}$. Then, $n^3 = (2k)^3 = 8k^3 = 2(4k^3)$. Since $4k^3 \in \mathbb{Z}, n^3$ is even.

($\Leftarrow$) WTS: n^3 is even $\implies$ n is even. We will prove the contrapositive: n is odd $\implies$ n^3 is odd. Suppose n is odd. Then, $n = 2k + 1$ *f.s.* $k \in \mathbb{Z}$. Then,

$$n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 8k + 1 = 2(4k^3 + 6k^2 + 4k) + 1.$$

Since $4k^3 + 6k^2 + 4k \in \mathbb{Z}, n^3$ is odd.

■

1.29 Chapter 1.4 # 20-b

`Let` n `be an integer. Prove that` n `is odd if and only if` n^3 `is odd.`

Proof 30.

($\Rightarrow$) WTS: n is odd $\implies$ n^3 is odd. This statement is previously proven.

($\Leftarrow$) WTS: n^3 is odd $\implies$ n is odd. We will prove the contrapositive: n is even $\implies$ n^3 is even. The contrapositive is also previously proven.

■

1.30 Chapter 1.4 # 21

`Prove that` $\sqrt[3]{2}$ `is irrational.`

Proof 31.

Assume for the sake of contradiction that $\sqrt[3]{2}$ is rational. Suppose $\sqrt[3]{2}$ is rational. By definition, $\exists p, q \in \mathbb{Z}$ *s.t.* $\sqrt[3]{2} = \frac{p}{q}$, assuming p and q have no common factors and $q \neq 0$. Raise the two sides of the equation to cube:

$$2 = \left(\frac{p}{q}\right)^3 = \frac{p^3}{q^3}.$$

Then, $p^3 = 2q^3$. Since $q^3 \in \mathbb{Z}$, we know p^3 is even. Then, p is also even (previously proven). Then, $p = 2k$ *f.s.* $k \in \mathbb{Z}$. Hence,

$$2q^3 = p^3 = (2k)^3 = 8k^3$$
$$q^3 = 4k^3 = 2(2k^3)$$

Since $2k^3 \in \mathbb{Z}$, we see q^3 is even. Then, q is also even.

⋇ This contradicts with our assumption that p and q have no common factors as p, q being even indicates they have 2 as their common factor.

So, our assumption is wrong, and $\sqrt[3]{2}$ is irrational.

∎

2 Sets

> *2.1 Handout Chapter 2.1 - Sets and Subsets*
>
> `Prove that` $\{12a + 4b \mid a, b \in \mathbb{Z}\} = \{4c \mid c \in \mathbb{Z}\}$.

Proof 1.

($\subseteq$) Suppose $x \in \{12a + 4b \mid a, b \in \mathbb{Z}\}$. Then, $x = 12a + 4b$ *f.s.* $a, b \in \mathbb{Z}$. So, $x = 12a + 4b = 4(3a + b)$. As $3a + b \in \mathbb{Z}$, we have $x \in \{4c \mid c \in \mathbb{Z}\}$. By definition, $\{12a + 4b \mid a, b \in \mathbb{Z}\} \subseteq \{4c \mid c \in \mathbb{Z}\}$.

($\supseteq$) Suppose $x \in \{4c \mid c \in \mathbb{Z}\}$. Then, $x = 4c$ *f.s.* $c \in \mathbb{Z}$. Suppose $c = 3a + b$ *f.s.* $a, b \in \mathbb{Z}$. Then, $x = 4c = 4(3a + b) = 12a + 4b$. By definition, $\{4c \mid c \in \mathbb{Z}\} \subseteq \{12a + 4b \mid a, b \in \mathbb{Z}\}$

Hence, we have proven $\{12a + 4b \mid a, b \in \mathbb{Z}\} = \{4c \mid c \in \mathbb{Z}\}$.

■

> *2.2 Exam 1 Review 2-m*
>
> `If` $A = \{x \mid x = n^4 - 1,\ \ n \in \mathbb{Z}\}$ `and` $B = \{x \mid x = m^2 - 1,\ \ m \in \mathbb{Z}\}$`, then` $A \subseteq B$.

Proof 2.

Suppose $x \in A$. Then, $x = n^4 - 1$ *f.s.* $n \in \mathbb{Z}$. Then, $x = n^4 - 1 = (n^2)^2 - 1$. Since $n^2 \in \mathbb{Z}$, we have $x \in B$. Therefore, $A \subseteq B$.

■

> *2.3 Exam 1 Review 2-n*
>
> `If` A`,` B`, and` C `are sets, then` $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof 3.

($\subseteq$) Suppose $x \in A \cap (B \cup C)$. WTS: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. By definition, $x \in A$ and $x \in (B \cup C)$. By definition, $x \in A$ and $x \in B$ or $x \in A$ and $x \in C$. Therefore, $x \in (A \cap B)$ or $x \in (A \cap C)$. That is, $x \in (A \cap B) \cup (A \cap C)$. Hence, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. □

($\supseteq$) Suppose $x \in (A \cap B) \cup (A \cap C)$. WTS: $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. By definition, $x \in (A \cap B)$ or $x \in (A \cap C)$. WLOG, consider $x \in (A \cap B)$. Then, $x \in A$ and $x \in B$. Similarly, we know $x \in A$ and $x \in C$ from $x \in (A \cap C)$. Therefore, $x \in A$ and $x \in B$ or $x \in C$. That is, $x \in A$ and $x \in (B \cup C)$, or $x \in A \cap (B \cup C)$. Hence, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

As $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$, we have shown that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

■

2.4 Exam 1 Review 2-o

For subsets A and B of a universal set U, $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Proof 4.

($\subseteq$) Suppose $x \in \overline{A \cup B}$. By definition, $x \notin A \cup B$. That is, $x \notin A$ and $x \notin B$. Or, $x \in \overline{A}$ and $x \in \overline{B}$. That is, $x \in \overline{A} \cap \overline{B}$. Therefore, $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$. □

($\supseteq$) Suppose $x \in \overline{A} \cap \overline{B}$. By definition, $x \notin A$ and $x \notin B$. That is, $x \in \overline{A \cup B}$. Therefore, $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

Since $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$, we have $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

■

2.5 Exam 1 Review 2-p

Suppose that A, B, and C are subsets of a universal set U. Let P and Q be the following statements:

P: $A \subseteq B$ or $A \subseteq C$; and

Q: $A \subseteq B \cap C$.

Write the statement $P \implies Q$, its converse, and its contrapositive. Prove the true ones or give counterexamples.

Claim. $P \implies Q : A \subseteq B$ or $A \subseteq C \implies A \subseteq B \cap C$.

Proof 5.

Suppose $x \in A$.

$\boxed{\text{Case 1}}$ Suppose $A \subseteq B$. Then $x \in B$. Since $B \cap C \subseteq B, x \in B \cap C$. Therefore, $A \subseteq B \cap C$.

$\boxed{\text{Case 2}}$ Suppose $A \subseteq C$. Then $x \in C$. Since $B \cap C \subseteq C, x \in B \cap C$. Therefore, $A \subseteq B \cap C$.

In both cases, we proven $A \subseteq B$ or $A \subseteq C \implies A \subseteq B \cap C$.

■

Claim. Converse: $Q \implies P$: $A \subseteq B \cap C \implies A \subseteq B$ or $A \subseteq C$.

Proof 6.

Suppose $A \subseteq B \cap C$. Suppose $x \in A$. Then $x \in B \cap C$. By definition, $x \in B$ and $x \in C$. Hence, $A \subseteq B$ and $A \subseteq C$. Since the "or" here is inclusive, $A \subseteq B$ and $A \subseteq C$ is a true case for $A \subseteq B$ or $A \subseteq C$. Hence, $A \subseteq B \cap C \implies A \subseteq B$ or $A \subseteq C$.

■

Claim. Contrapositive: $\neg Q \implies \neg P$: $A \nsubseteq B \cap C \implies A \nsubseteq B$ and $A \nsubseteq C$.

Proof 7.

Since the original statement is true, its contrapositive is automatically true.

■

2.6 Handout Chapter 2.2 # 10-a-i

`Let` $A = \{6a + 4 \mid x \in \mathbb{Z}\}$ `and` $B = \{18b - a \mid b \in \mathbb{Z}\}$. `Prove or disprove:` $A \subseteq B$.

Disproof 8.

Suppose $x \in A$. Then $x = 6a + 3$ *f.s.* $a \in \mathbb{Z}$. Notice that $6a + 4 = 18\left(\frac{1}{3}a + \frac{1}{3}\right) - 2$. Since $\frac{1}{3}a + \frac{1}{3} = \frac{1}{3}(a+1) \in \mathbb{Q}$, but $\frac{1}{3}(a+1) \notin \mathbb{Z} \forall a \in \mathbb{Z}$, we have $6a + 4 \notin \{18b - 2 \mid b \in \mathbb{Z}\}$. By definition of subsets, $A \nsubseteq B$.

Remark *We can also use proof by contradiction to disprove this statement.*

■

2.7 Handout Chapter 2.2 # 10-a-ii

`Let` $A = \{6a + 4 \mid x \in \mathbb{Z}\}$ `and` $B = \{18b - a \mid b \in \mathbb{Z}\}$. `Prove or disprove:` $B \subseteq A$.

Proof 9.

Suppose $x \in B$. Then, $x = 18b - 2$ *f.s.* $b \in \mathbb{Z}$. Notice that $18b - 2 = 6(3b - 1) + 4$. Since $3b - 1 \in \mathbb{Z}$, we have $x \in A$. Hence, by definition of subsets, $B \subseteq A$.

■

2.8 Handout Chapter 2.2 # 10-b

`If` A `and` B `are sets, then` $\mathcal{P}(A) - \mathcal{P}(B) = \mathcal{P}(A - B)$.

Proof 10.

($\subseteq$) WTS: $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$. Suppose $X \in \mathcal{P}(A) - \mathcal{P}(B)$. By definition of set difference, $X \in \mathcal{P}(A)$ and $X \notin \mathcal{P}(B)$. By definition of power sets, $X \subseteq A$ and $X \nsubseteq B$. Hence, $X \subseteq (A - B)$, by definition of set difference. Therefore, $X \in \mathcal{P}(A - B)$, and thus $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$ as desired. □

($\supseteq$) WTS: $\mathcal{P}(A - B) \subseteq \mathcal{P}(A) - \mathcal{P}(B)$. Suppose $X \in \mathcal{P}(A - B)$. Then, $X \subseteq A - B$. By definition of set difference, $X \subseteq A$ and $X \nsubseteq B$. Then, $X \in \mathcal{P}(A)$ and $X \notin \mathcal{P}(B)$. By definition of set difference, $X \in \mathcal{P}(A) - \mathcal{P}(B)$. Hence, $\mathcal{P}(A - B) \subseteq \mathcal{P}(A) - \mathcal{P}(B)$.

■

2.9 *Handout Chapter 2.2 # 10-c*

`If` A`,` B`, and` C `are sets, and` $A \times B = B \times C$`, then` $A = B$`.`

Proof 11.

Suppose A, B, and C are sets. Suppose $\exists a, b \in \mathbb{Z}$ *s.t.* $(a, c) \in A \times C$. By definition of Cartesian product, $a \in A$ and $c \in C$. Suppose $\exists b, c \in \mathbb{Z}$ *s.t.* $(b, c) \in B \times C$. So, we know that $b \in B$. Suppose $A \times C = B \times C$. Then, $A \times C \subseteq B \times C$ and $A \times C \supseteq B \times C$.

($\subseteq$) If $A \times C \subseteq B \times C$, we have $(a, c) \in B \times C$. Then, $a \in B$. Since $a \in A$, we know $A \subseteq B$. □

($\supseteq$) Similarly, since $A \times C \supseteq B \times C$, we have $(b, c) \in A \times C$. Then, $b \in A$. Since $b \in B$, we see that $B \subseteq A$.

By definition of set equality, $A = B$.

■

2.10 *Chapter 2.1 # 6*

`Let` $n \in \mathbb{Z}$ `and let` $A = n\mathbb{Z}$`. Prove that if` $x, y \in A$`, then` $x + y \in Z$ `and` $xy \in A$`.`

Proof 12.

Suppose $n \in \mathbb{Z}$ and $A = n\mathbb{Z}$. Then, $A = \{nk \mid k \in \mathbb{Z}\}$. Suppose $x, y \in A$. Then, $\exists k, l$ *s.t.* $x = nk$ and $y = nl$. Then, $x + y = nk + nl = n(k + l)$. Since $k + l \in \mathbb{Z}, x + y \in A$. Similarly, $xy = (nk)(nl) = n(nkl)$. Since $nkl \in \mathbb{Z}, xy \in A$.

■

2.11 *Chapter 2.1 # 10*

`Let` n `and` m `be integers. Let` $A = n\mathbb{Z}$ `and` $B = m\mathbb{Z}$`. Prove that if` n `is a multiplier of` m`, then` $A \subseteq B$`.`

Proof 13.

Let n and m be integers. Let $A = n\mathbb{Z}$ and $B = m\mathbb{Z}$. Suppose $x \in A$. Then, by definition, $\exists k \in \mathbb{Z}$ *s.t.* $x = nk$. Since n is a multiplier of m, $n = ml$ *f.s.* $l \in \mathbb{Z}$. Then, $x = nk = (ml)k = m(lk)$. Since $lk \in \mathbb{Z}$, $x = m(lk)$ is a multiplier of m. That is, $x \in m\mathbb{Z}$. Hence, $A \subseteq B$.

■

> *2.12 Chapter 2.1 # 12*
>
> Let $A = \{n \in \mathbb{Z} \mid n \text{ is a multiple of } 4\}$ and $B = \{n \in \mathbb{Z} \mid n^2 \text{ is a multiple of } 4\}$. Prove that $A \subseteq B$ and $B \nsubseteq A$.

Proof 14.

WTS: $A \subseteq B$. Suppose $x \in A$. Then, $\exists k \in \mathbb{Z}$ *s.t.* $x = 4k$. Consider $x^2 = (4k)^2 = 16k^2 = 4(8k^2)$. Since $8k^2 \in \mathbb{Z}$, by definition of divides, x^2 is a multiple of 4. Hence, by definition of set B, $x \in B$. That is, $A \subseteq B$.

■

Proof 15.

WTS: $B \nsubseteq A$. Consider $x = 2k$ *f.s.* $k \in \mathbb{Z}$. Then, $x^2 = (2k)^2 = 4k^2$. Since $k^2 \in \mathbb{Z}$, x^2 is a multiple of 4. Hence, $x \in B$. However, $x = 2k$ is not a multiple of 4. That is, $x \notin A$. Hence, we found an element of B that is not an element of A. Then, by definition, $B \nsubseteq A$.

■

> *2.13 Chapter 2.1 # 13*
>
> If $A = \{n \in \mathbb{Z} \mid n + 3 \text{ is odd}\}$, then A is equal to the set of all even integers.

Proof 16.

Suppose $B = \{n \in \mathbb{Z} \mid n \text{ is even}\}$. Then, B is the set of all even numbers.

($\subseteq$) Suppose $x \in A$. Then, by definition, $x + 3$ is odd. That is, $\exists k \in \mathbb{Z}$ *s.t.* $x + 3 = 2k + 1$. Then, $x = 2k + 1 - 3 = 2k - 2 = 2(k - 1)$. Since $k - 1 \in \mathbb{Z}$, then x is even. Therefore, $x \in B$, and $A \subseteq B$. □

($\supseteq$) Suppose $x \in B$. Then, x is even. So, $\exists k \in \mathbb{Z}$ *s.t.* $x = 2k$. Consider $x + 3 = 2k + 3 = 2k + 2 = 1 = 2(k + 1) + 1$. Since $k + 1 \in \mathbb{Z}$, then $x + 3$ is odd. Hence, $x \in A$, and $B \subseteq A$.

Collectively, we've proven $A = B$.

■

> *2.14 Chapter 2.1 # 15*
>
> Let $A = \{n \in \mathbb{Z} \mid n = 4t + 1 \text{ for some } t \in \mathbb{Z}\}$ and $B = \{n \in \mathbb{Z} \mid n = 4t + 9 \text{ for some } t \in \mathbb{Z}\}$. Prove that $A = B$.

Proof 17.

($\subseteq$) Suppose $x \in A$. Then, $x = 4t + 1$ *f.s.* $t \in \mathbb{Z}$. Note that $x = 4t + 9 - 8 = (4t - 8) + 9 = 4(t - 2) + 9$. Since $t - 2 \in \mathbb{Z}$, by definition, $x \in B$. Then, $A \subseteq B$. □

($\supseteq$) Suppose $x \in B$. Then, $x = 4t + 9$ *f.s.* $t \in \mathbb{Z}$. Note that $x = 4t + 9 = 4t + 8 + 1 = 4(t + 2) + 1$. Since $t + 2 \in \mathbb{Z}$, by definition, $x \in A$. Hence, $B \subseteq A$.

Collectively, we've proven $A = B$.

■

2.15 Chapter 2.1 # 16

`Let` $A = \{n \in \mathbb{Z} \mid n = 3t + 1$ `for some` $t \in \mathbb{Z}\}$ `and` $B = \{n \in \mathbb{Z} \mid n = 3t + 2$ `for some` $t \in \mathbb{Z}\}$`.` `Prove that` A `and` B `have no elements in common.`

Proof 18.

Assume for the sake of contradiction that A and B have one element in common, and suppose that element is x. By our assumption, $x \in A$. So, $x = 3t + 1$ *f.s.* $t \in \mathbb{Z}$. Also, $x \in B$, so $x = 3s + 2$ *f.s.* $s \in \mathbb{Z}$. Then, we have $x = 3t + 1 = 3s + 2$. Solve for t, we have

$$3t = 3s + 2 - 1 = 3s + 1$$
$$t = \frac{3s + 1}{3} = s + \frac{1}{3}$$

Since $s \in \mathbb{Z}$, $\frac{1}{3} \notin \mathbb{Z}$, we have $t = s + \frac{1}{3} \notin \mathbb{Z}$.

⋇ This contradicts with the fact that $t \in \mathbb{Z}$.

So, our assumption is wrong, and A and B have no elements in common.

■

2.16 Chapter 2.3 # 8

`Let` $A_i = (-i, i) = \{x \in \mathbb{R} \mid -i < x < i\}$`.` `Prove that` $\bigcup_{i=1}^{\infty}(-i, i) = \mathbb{R}$ `and` $\bigcap_{i=1}^{\infty}(-i, i) = (-1, 1)$.

Proof 19.

WTS: $\bigcup_{i=1}^{\infty}(-i, i) = \mathbb{R}$

($\subseteq$) Suppose for some $k \in \mathbb{Z}$ and $k \geq 1$, $x \in A_k$. That is, $x \in (-k, k)$. Since $k \geq 1$, by definition of union, $A_k \subseteq \bigcup_{i=1}^{\infty}(-i, i)$. Hence, $x \in \bigcup_{i=1}^{\infty}(-i, i)$. Since $A_k \subseteq \mathbb{R}$, $x \in \mathbb{R}$. Hence, $\bigcup_{i=1}^{\infty}(-i, i) \subseteq \mathbb{R}$. □.

($\supseteq$) Suppose $x \in \mathbb{R}$. Consider the set $(-k, k) = A_k$, where $k \in \mathbb{Z}$ and $k \geq x$. Then, $x \in (-k, k)$. Since $k \in \mathbb{Z}$, then $A_k \subseteq \bigcup_{i=1}^{\infty}(-i, i)$ by definition of union. Then, $x \in \bigcup_{i=1}^{\infty}(-i, i)$. That is, $\mathbb{R} \subseteq \bigcup_{i=1}^{\infty}(-i, i)$.

■

Proof 20.

WTS: $\bigcap_{i=1}^{\infty}(-i,i) = (-1,1)$.

($\subseteq$) Let $x \in \bigcap_{i=1}^{\infty}(-i,i)$. So, $x \in A_i \quad \forall i = \{1,2,3,\cdots\}$. Specially, $x \in A_1 = (-1,1)$. Hence, $\bigcap_{i=1}^{\infty}(-i,i) \subseteq (-1,1)$. $\square$

($\supseteq$) Let $x \in (-1,1)$. Let $k \in \{1,2,3,\cdots\}$. We will show $x \in A_k$. Since $k \geq 1$, then $-k \leq -1$. Form $x \in (-1,1)$, we know $-1 < x < 1$. Then, $-k \leq -1 < x < 1 \leq k$. That is, $-k < x < k$, or $x \in (-k,k) = A_k$. Since k is arbitrary, we've proven $x \in A_k \quad \forall k \geq 1$. So, $x \in \bigcap_{i=1}^{\infty}(-i,i)$. Hence, $(-1,1) \subseteq \bigcap_{i=1}^{\infty}(-i,i)$.

$\blacksquare$

2.17 Chapter 2.3 # 10

`Let` $A_i = \{1,2,3,\cdots,i\}$ `for` $i \in \mathbb{Z}^+$. `Compute` $\bigcup_{i=1}^{\infty} A_i$ `and` $\bigcap_{i=1}^{\infty} A_i$. `Prove your answer.`

Claim. $\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}^+$.

Proof 21.

($\subseteq$) Let $x \in \bigcup_{i=1}^{\infty} A_i$. Then $x \in A_k$ *f.s.* $k \in \mathbb{Z}^+$. That is, by definition, $x \in \{1,2,3,\cdots,k\}$. Since $k \in \mathbb{Z}^+$, $\{1,2,3,\cdots,k\} \subseteq \mathbb{Z}^+$, $x \in \mathbb{Z}^+$. $\square$

($\supseteq$) Let $x \in \mathbb{Z}^+$. Consider $A_{x+1} = \{1,2,3,\cdots,x+1\}$. Then, $x \in A_{x+1}$. By definition of union, $A_{x+1} \subseteq \bigcup_{i=1}^{\infty} A_i$. So, $x \in \bigcup_{i=1}^{\infty} A_i$.

Hence, we've shown $\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}^+$.

$\blacksquare$

Claim. $\bigcap_{i=1}^{\infty} A_i = \{1\}$.

Proof 22.

($\subseteq$) Suppose $x \in \bigcap_{i=1}^{\infty} A_i$. By definition of union, $x \in A_k \quad \forall k \geq 1$. Specially, $x \in A_1 = \{1\}$. $\square$

($\supseteq$) Suppose $x \in \{1\}$. Let $k \geq 1$. By definition, $A_k = \{1,2,3,\cdots.k\}$. Since $\{1\} \subseteq \{1,2,3,\cdots,k\} = A_k$, $x \in A_k$. As k was arbitrary, we've proven $x \in A_k \quad \forall k \geq 1$. So, $x \in \bigcap_{i=1}^{\infty} A_i$. Hence, $\{1\} \subseteq \bigcap_{i=1}^{\infty} A_i$.

$\blacksquare$

2.18 Chapter 2.3 # 10

`Let` $A_i = [i,i+1) = \{x \in \mathbb{R} \mid i \leq x < i+1\}$ `for` $i \in \mathbb{Z}^+$. `Compute` $\bigcup_{i=1}^{\infty} A_i$ `and` $\bigcap_{i=1}^{\infty} A_i$. `Prove your answer.`

Claim. $\bigcup_{i=1}^{\infty} A_i = \{x \in \mathbb{R} \mid x \geq 1\}$.

Proof 23.

($\subseteq$) Suppose $x \in \bigcup_{i=1}^{\infty} A_i$. By definition of union, $x \in A_k$ *f.s.* $k \in \{1, 2, \cdots\}$. By definition, $A_k = [k, k+1)$, so $k \leq x < k+1$. Since $k \geq 1$, we have $1 \leq k \leq x < k+1$. That is, $x \in \{x \in \mathbb{R} \mid x \geq 1\}$. Hence, $\bigcup_{i=1}^{\infty} A_i \subseteq \{x \in \mathbb{R} \mid x \geq 1\}$. $\square$

($\supseteq$) Suppose $x \in \{x \in \mathbb{R} \mid x \geq 1\}$. Then, $x \geq 1$. Consider $A_x = [x, x+1)$, we have $x \in [x, x+1)$. By definition of union, $A_x \subseteq \bigcup_{i=1}^{\infty} A_i$. Hence, $x \in \bigcup_{i=1}^{\infty} A_i$, or $\{x \in \mathbb{R} \mid x \geq 1\} \subseteq \bigcup_{i=1}^{\infty} A_i$.

$\blacksquare$

Claim. $\bigcap_{i=1}^{\infty} A_i = \varnothing$.

Proof 24.

Note that $n+1 \in A_{n+1}$. However, $n+1 \notin A_n = [n, n+1)$. That is, for every $n \in \mathbb{Z}^+$, $n+1$ is not in every A_i. So, by definition of set intersection, $\bigcap_{i=1}^{\infty} A_i = \varnothing$.

$\blacksquare$

2.19 Chapter 2.3 # 12

`Let` $A_i = \left(\frac{1}{i}, i\right] = \left\{x \in \mathbb{R} \mid \frac{1}{i} < x \leq i\right\}$ `for` $i \geq 2$`. Compute` $\bigcup_{i=1}^{\infty} A_i$ `and` $\bigcap_{i=1}^{\infty} A_i$`. Prove your answer.`

Claim. $\bigcup_{i=1}^{\infty} A_i = (0, \infty)$.

Proof 25.

($\subseteq$) Suppose $x \in \bigcup_{i=1}^{\infty} A_i$. Then, $x \in A_k$ *f.s.* $k \geq 2$. By definition of A_i, $x \in A_k = \left(\frac{1}{k}, k\right]$. Since $\left(\frac{1}{k}, k\right] \subseteq (0, \infty)$, we know $x \in (0, \infty)$. $\square$

($\supseteq$) Suppose $x \in (0, \infty)$. Consider $\lceil x \rceil$, the minimum integer greater than x. Suppose $k = \lceil x \rceil$, then $A_k = \left(\frac{1}{k}, k\right]$. Since $k \geq x$, by definition of the ceiling function, $x \in A_k$. Since $A_k \subseteq \bigcup_{i=1}^{\infty} A_i$, we know that $x \in \bigcup_{i=1}^{\infty} A_i$.

$\blacksquare$

Claim. $\bigcap_{i=1}^{\infty} A_i = \left(\frac{1}{2}, 2\right]$.

Proof 26.

($\subseteq$) Suppose $x \in \bigcap_{i=1}^{\infty} A_i$. Then, $x \in A_k \quad \forall k \geq 2$. Specially, $x \in A_2 = \left(\frac{1}{2}, 2\right]$. $\square$

($\supseteq$) Suppose $x \in \left(\frac{1}{2}, 2\right]$. Consider $A_k = \left(\frac{1}{k}, k\right]$ *f.s.* $k \geq 2$. Since $k \geq 2, \frac{1}{2} \leq \frac{1}{2}$. Then, $\left(\frac{1}{2}, 2\right] \subseteq \left(\frac{1}{k}, k\right]$. Hence, $x \in A_k$. Since k is arbitrary, we have proven that $x \in A_k \quad \forall k \geq 2$. That is, $x \in \bigcap_{i=1}^{\infty} A_i$.

■

> *2.20 Chapter 2.3 # 13*
> `Let` $A_i = \left[i, 1 + \frac{1}{i}\right]$ `for` $i \in \mathbb{Z}^+$. `Compute` $\bigcup_{i=1}^{\infty} A_i$ `and` $\bigcap_{i=1}^{\infty} A_i$. `Prove your answer.`

Claim. $\bigcup_{i=1}^{\infty} A_i = [1, 2]$.

Proof 27.

($\subseteq$) Suppose $x \in \bigcup_{i=1}^{\infty} A_i$. Then, $x \in A_k$ *f.s.* $k \in \mathbb{Z}^+$. Hence, $x \in A_k = \left[1, 1 + \frac{1}{k}\right]$. That is, $1 \leq x \leq 1 + \frac{1}{k}$. Since $k \in \mathbb{Z}^+$, $\frac{1}{k} \leq 1$. Then, $1 + \frac{1}{k} \leq 2$. So, $1 \leq x \leq 1 + \frac{1}{2} \leq 2$, or $x \in [1, 2]$. □

($\supseteq$) Suppose $x \in [1, 2]$. Note that $A_1 = [1, 2]$, so $x \in A_1$. Since $A_1 \subseteq \bigcup_{i=1}^{\infty} A_i$, by definition of set union, $x \in \bigcup_{i=1}^{\infty} A_i$.

■

Claim. $\bigcap_{i=1}^{\infty} A_i = \{1\}$.

Proof 28.

($\subseteq$) Suppose $x \in \bigcap_{i=1}^{\infty} A_i$. Then, $x \in A_k \quad \forall k \in \mathbb{Z}^+$. By definition of A_k, $x \in A_k = \left[1, 1 + \frac{1}{k}\right]$. Note $\lim_{k \to 0} \left(1 + \frac{1}{k}\right) = 1 + 0 = 1$. So, $A_k = [1, 1] = \{1\}$, when $k \to \infty$. □

($\supseteq$) Suppose $x \in \{1\}$. Consider $A_k = \left[1, 1 + \frac{1}{k}\right]$ for some $k \in \mathbb{Z}^+$. Since $1 \in \left[1, 1 + \frac{1}{k}\right]$, we have $x \in \left[1, 1 + \frac{1}{k}\right] = A_k$. Since k is arbitrary, $x \in A_k \quad \forall k \in \mathbb{Z}^+$. That is, $x \in \bigcap_{i=1}^{\infty} A_i$.

■

> *2.21 Chapter 2.3 # 14*
> `Let` $A_i = \left(i, 1 + \frac{1}{i}\right)$ `for` $i \in \mathbb{Z}^+$. `Compute` $\bigcup_{i=1}^{\infty} A_i$ `and` $\bigcap_{i=1}^{\infty} A_i$. `Prove your answer.`

Claim. $\bigcup_{i=1}^{\infty} A_i = (1, 2)$, and $\bigcap_{i=1}^{\infty} A_i = \varnothing$.

Proof 29.

Similar proofs as done in the previous exercise.

■

2.22 Exam 2 Review 2

`For sets` A, B, C, D, `prove that` $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Proof 30.

Let A, B, C, D be sets.

($\subseteq$) Suppose $(x, y) \in (A \times B) \cap (C \times D)$. By definition of set intersection, $(x, y) \in A \times B$ and $(x, y) \in C \times D$. Since $(x, y) \in A \times B$, by definition of Cartesian product, $x \in A$ and $y \in B$. Similarly, since $(x, y) \in C \times D$, $x \in C$ and $y \in D$. Since $x \in A$ and $x \in C$, by definition of set intersection, $x \in A \cap C$. Similarly, since $y \in B$ and $y \in D$, $y \in B \cap D$. Hence, $(x, y) \in (A \cap C) \times (B \cap D)$, by definition of Cartesian product.

($\supseteq$) Suppose $(x, y) \in (A \cap C) \times (B \cap D)$. By definition of Cartesian product, $x \in A \cap C$ and $y \in B \cap D$. Since $x \in A \cap C$, by definition of set intersection, $x \in A$ and $x \in C$. Similarly, since $y \in B \cap D$, $y \in B$ and $y \in D$. Note that $x \in A$ and $y \in B$. Hence, $(x, y) \in A \times B$. Further, since $x \in C$ and $y \in D$, $(x, y) \in C \times D$. Therefore, $(x, y) \in A \times B$ and $(x, y) \in C \times D$. By definition of set intersection, $(x, y) \in (A \times B) \cap (C \times D)$.

■

2.23 Exam 2 Review 3

`Given the indexed sets, compute the unions and intersections. Give full and careful proofs of each:` $A_i = [i-1, i]$ `for` $i = 1, \cdots, n$. `Compute` $\bigcup_{i=1}^{n} A_i$ `and` $\bigcap_{i=1}^{n} A_i$.

Claim. $\bigcap_{i=1}^{n} A_i = \begin{cases} A_1, & n = 1 \\ A_1 \cap A_2 = \{1\}, & n = 2 \\ \varnothing, & n \geq 3 \end{cases}$.

Proof 31.

We will prove that if $n \geq 3$, $\bigcap_{i=1}^{n} A_i = \varnothing$. Suppose $x \in A_k$ *f.s.* $k \in \{1, 2, 3, \cdots, n\}$. Then, by definition, $k - 1 \leq x \leq k$. Consider $A_{k+2} = [k+1, k+2]$. Since $k < k+1$, $x \notin [k+1, k+2]$. Hence, $\bigcap_{i=1}^{n} A_i = \varnothing$.

■

Proof 32.

Alternatively, we can use proof by contradiction. Suppose $n \geq 3$. Assume for the sake of contradiction that $\bigcap_{i=1}^{n} A_i \neq \varnothing$. Then, $\exists\, x \in \bigcap_{i=1}^{n} A_i$. So, $x \in A_i \quad \forall i \in \{1, 2, 3, \cdots .n\}$. Since $n \geq 3$, specifically,

$x \in A_1 = [0,1]$ and $x \in A_3 = [2,3]$. ⋇ But this is a contradiction because $A_1 \cap A_3 = \varnothing$. So, it must be that $\bigcap\limits_{i=1}^{n} A_i = \varnothing$.

■

Claim. $\bigcup\limits_{i=1}^{n} A_i = [0,n]$.

Proof 33.

($\subseteq$) Suppose $x \in \bigcup\limits_{i=1}^{n} A_i$. Then, $x \in A_k$ *f.s.* $k \in \{1, 2, \cdots, n\}$. Then, by definition of A_i, $x \in [k-1,k]$, or $k-1 \leq x \leq k$. Since $1 \leq k \leq n$ and $0 \leq k-1 \leq n-1$, we have $0 \leq k-1 \leq x \leq k \leq n$. So, $x \in [0,n]$. □

($\supseteq$) Let $x \in [0,n]$.

$\boxed{\text{Case 1}}$ $x = 0$. Note that $x \in [0,1] = A_1$. Then, $x \in \bigcup\limits_{i=1}^{n} A_i$.

$\boxed{\text{Case 2}}$ When $x > 0$, set $k = \lceil x \rceil$. Then, $k \in \mathbb{N}$ and $1 \leq k \leq n$. Then, $k-1 \leq x \leq k$. That is, $x \in [k-1,k] = A_k$. So, $x \in \bigcup\limits_{i=1}^{n} A_i$.

■

> *2.24 Exam 2 Review 4*
>
> `Here's a mathematical statement:`
>
> (s): `for all sets` A `and` B, $A \subseteq B$ `implies that` $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.
>
> `State the converse (`s_1`) of (`s`), the contrapositive (`s_2`) of (`s`), the negation (`$\neg s$`) of (`s`). Which of the statements (`s`), (`s_1`), (`s_2`), (`$\neg s$`) are true?`

Claim. (s) is true.

Proof 34.

Let A and B be sets. Suppose $A \subseteq B$. Suppose $X \subseteq A$. Since $A \subseteq B, X \subseteq B$. Because $X \subseteq A$, $X \in \mathcal{P}(A)$. Since $X \subseteq B, X \in \mathcal{P}(B)$. Therefore, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

■

Claim. (s_1): "for all sets A and B, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ implies $A \subseteq B$" is true.

Proof 35.

Let A and B be sets. Suppose $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Suppose $X \in \mathcal{P}(A)$. Then, $X \subseteq A$. By definition of subsets, $X \in \mathcal{P}(B)$. So, $X \subseteq B$. Suppose $x \in X$. Since $X \subseteq A$, $x \in A$. Similarly, since $X \subseteq B$, $x \in B$. Therefore, $A \subseteq B$.

■

Claim. Since (s) is true, the contrapositive of it (s_2), "for all sets A and B, $\mathcal{P}(A) \nsubseteq \mathcal{P}(B)$ implies $A \nsubseteq B$," will be true for sure.

Claim. Since (s) is true, the negation of it $(\neg s)$ "for all sets A and B, $A \subseteq B$ and $\mathcal{P}(A) \nsubseteq \mathcal{P}(B)$," will be false.

2.25 Exam 2 Review 5

`For all sets` A `and` B, `if` $\mathcal{P}(A) = \mathcal{P}(B)$, `then` $A = B$.

Proof 36.

To prove set equality, we will prove $A \subseteq B$ and $B \subseteq A$. However, since A and B are symmetric, WLOG, proving $A \subseteq B$ is sufficient. Suppose $X \in \mathcal{P}(A)$. Then, $X \subseteq A$. Since $\mathcal{P}(A) = \mathcal{P}(B)$, $X \in \mathcal{P}(B)$. So, $X \subseteq B$. Suppose $x \in X$. Since $X \subseteq A$, $x \in A$. Similarly, since $X \subseteq B$, $x \in B$. Therefore, for all $x \in A$, $x \in B$. By definition of subset, $A \subseteq B$.

■

2.26 Exam 2 Review 7

`Find` $\bigcap_{n \in \mathbb{N}} = n\mathbb{Z}$.

Claim. $\bigcap_{n \in \mathbb{N}} = n\mathbb{Z} = \{0\}$.

Proof 37.

$(\subseteq)$ WTS: $0 \in n\mathbb{Z} \quad \forall n \in \mathbb{N}$. Let $n \in \mathbb{N}$. Consider $n\mathbb{Z}$. Note that $0 = n(0)$. Since $0 \in \mathbb{Z}, 0 \in n\mathbb{Z}$. Since we picked an arbitrary $n \in \mathbb{Z}$, we've shown that $0 \in n\mathbb{Z} \quad \forall n \in \mathbb{N}$. By definition of intersection, $0 \in \bigcap_{n \in \mathbb{N}} = n\mathbb{Z}$. So, $\{0\} \subseteq \bigcap_{n \in \mathbb{N}} = n\mathbb{Z}$. □

$(\supseteq)$ Suppose for the sake of contradiction that an integer $\neq 0$ belongs to the intersection. Then, $\exists x \neq 0$ ***s.t.*** $x \in n\mathbb{Z} \quad \forall n \in \mathbb{N}$.

$\boxed{\text{Case 1}}$ If $x > 0$, then $x \in \mathbb{N}$. So, $2x \in \mathbb{N}$. Therefore, by our assumption, $x \in 2x\mathbb{Z}$. Then, $\exists k \in \mathbb{Z}$ ***s.t.*** $x = 2xk$. So, we get $k = \dfrac{x}{2x} = \dfrac{1}{2}$ since $x =\neq 0$. ⋇ This contradicts with the fact that $k \in \mathbb{Z}$. Therefore, our assumption is wrong. Hence, $\nexists x \neq 0$ ***s.t.*** $x \in n\mathbb{Z} \quad \forall n \in \mathbb{N}$.

$\boxed{\text{Case 2}}$ If $x < 0$, then $-x \in \mathbb{N}$. So, $-2x \in \mathbb{N}$. Therefore, by our assumption, $x \in -2x\mathbb{Z}$. Then, $\exists k \in \mathbb{Z}$ ***s.t.*** $x = -2xk$. So, we get $k = \dfrac{x}{-2x} = -\dfrac{1}{2}$ since $x \neq 0$. However, $k = \dfrac{1}{2} \notin \mathbb{Z}$. ⋇ This contradicts with the fact that $k \in \mathbb{Z}$. Therefore, our assumption is wrong. $\nexists x \neq 0$ ***s.t.*** $x \in n\mathbb{Z} \quad \forall n \in \mathbb{N}$.

■

3 Integers and Induction

3.1 Handout Chapter 5.1-5.2-Axioms of Integers

`Let` $a, b \in \mathbb{Z}$. `Then` $(-a)(-b) = ab$.

Proof 1.

Notice that $a \cdot 0 = 0$. Multiply (-1) on both sides:

$$(-a \cdot 0) = -0 = 0$$
$$(-a) \cdot 0 = 0$$

By additive identity, $b + (-b) = 0$, so we know that

$$(-a)(b + (-b)) = 0.$$

By distributivity,

$$(-a)b + (-a)(-b) = 0.$$

Add the additive inverse of $-ab$ to both sides:

$$-ab + (-(-ab)) + (-a)(-b) = 0 + (-(-ab))$$
$$0 + (-a)(-b) = 0 + ab$$
$$(-a)(-b) = ab.$$

■

3.2 Chapter 5.1 # 1-a

$-(-a) = a$ `for all` $a \in \mathbb{Z}$.

Proof 2.

By additive inverse, we know $a + (-a) = 0$. Multiply (-1) on both sides:

$$(-1)(a + (-a)) = 0$$
$$(-1)a + (-1)(-a) = 0 \qquad \textit{distributivity}$$

Add (a) on both sides, we get

$$\begin{aligned}
(-1)a + (-1)(-a) + a &= 0 + a \\
(-1)a + a + (-1)(-a) &= a && \textit{additive identity, commutativity} \\
-a + a + (-1)(-a) &= a \\
0 + (-1)(-a) &= a && \textit{additive inverse} \\
(-1)(-a) &= a && \textit{additive identity} \\
-(-a) &= a
\end{aligned}$$

■

3.3 Chapter 5.1 # 1-c

$a(b-c) = ab - ac$ for all $a, b, c \in \mathbb{Z}$.

Proof 3.

By distributivity,

$$\begin{aligned}
(b + (-c))a &= ba + (-c)a \\
&= ab + (-1)ac && \textit{commutativity} \\
&= ab - ac
\end{aligned}$$

■

3.4 Chapter 5.1 # 2

Let $a, b \in \mathbb{Z}$. Prove that $-(a+b) = -a - b$.

Proof 4.

$$\begin{aligned}
-(a+b) = (-1)(a+b) &= (-1)a + (-1)b && \textit{distributivity} \\
&= -a - b.
\end{aligned}$$

■

3.5 Chapter 5.1 # 3

Let $a, b \in \mathbb{Z}$. Suppose that $a < b$. Prove that $(-a) > (-b)$.

Proof 5.

By definition, we know that $a - b \in \mathbb{Z}^+$. Since $a - b = a + (-b) = (-b) + a = (-b) - (-a)$, we know $(-b) - (-a) \in \mathbb{Z}^+$. By definition, $(-b) < (-a)$. That is, $(-a) > (-b)$.

■

Theorem 3.1 (Well Ordering Principle for $\mathbb{N}$.) *If $X \subseteq \mathbb{N}$ and $X \neq \varnothing$, then $\exists x_0 \in X$* s.t. *$\forall a \in X$ and $a \neq x_0$, we have $a - x_0 \in \mathbb{Z}^+$.*

3.6 Exam 2 Review 6-a

`Every non-empty subset of the rational numbers` $\mathbb{Q}$ `contains a minimum element.`

Counterexample6.

Consider $(-\infty, 0) \cap \mathbb{Q}$. There will not be a minimum rational number in it.

□

Counterexample7.

Consider $(0, 1) \cap \mathbb{Q}$. There will not be a minimum element in it.

□

Proof 8.

Suppose $\exists\ s_0$ *s.t.* s_0 is the minimum element of $(0, 1) \cap \mathbb{Q}$. Since $s_0 \in \mathbb{Q}, \exists\ p, q \in \mathbb{Z}$ *s.t.* $s_0 = \frac{p}{q}$. Consider $\frac{p}{q+1}$. Since $1 \in \mathbb{Z}$, $q + 1 \in \mathbb{Z}$, then $\frac{p}{q+1} \in \mathbb{Q}$. Since $s_0 \in (0, 1)$ and s_0 is the minimum element of $(0, 1) \cap \mathbb{Q}$, $0 < s_0 < 1$ and there is no element between 0 and s_0. Then, $\frac{p}{q} > 0$. That means, $p \neq 0$. So, $\frac{p}{q+1} > 0$ as well. However, since $q + 1 > q$, $\frac{p}{q+1} < \frac{p}{q}$. That is, $\frac{p}{q+1} \in (0, s_0)$. ⋇ This contradicts with our assumption that there is no element in $(0, s_0)$. Hence, our assumption is incorrect. So, there is no minimum element of $(0, 1) \cap \mathbb{Q}$.

■

3.7 Exam 2 Review 8

`Prove that for all` $n \in \mathbb{N}$,

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n + 1) = \frac{n(n + 1)(n + 2)}{3}.$$

Proof 9.

Let $P(n)$ be the statement that "$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \frac{n(n + 1)(n + 2)}{3}$."

$\boxed{\text{Base Case}}$ Consider $P(1)$: $1 \cdot 2 = \frac{1(1+1)(1+2)}{3}$. Note that $1 \cdot 2 = 2$ and $\frac{1(1+1)(1+2)}{3} = \frac{1(2)(3)}{3} = 2$. Therefore, $1 \cdot 2 = \frac{1(1+1)(1+2)}{3}$. That is, $P(1)$ is correct.

$\boxed{\text{Inductive Steps}}$ Suppose $P(k)$ is true for some $k \in \mathbb{N}$. That is,

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3} \qquad ①$$

Add $(k+1)(k+2)$ on both sides of equation ①, we get

$$\begin{aligned} 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 \cdots + k(k+1) + (k+1)(k+2) &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} \\ &= \frac{(k+1)(k+2)(k+3)}{3}. \end{aligned}$$

Therefore, $P(k+1)$ is true given $P(k)$ is true.

Since we've proven that $P(1)$ is true and $P(k) \implies P(k+1)$, by Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$.

■

Definition 3.1 (Fibonacci Sequence) *The Fibonacci Sequence f_n is defined recursively as follows:*

$$f_1 = 1, \quad f_2 = 1, \quad \textit{and} \quad f_n = f_{n-1} + f_{n-2} \textit{ for } n \geq 3.$$

3.8 Exam 2 Review 9

`Prove that for all` $n \in \mathbb{N}$,

$$f_{n+1}^2 - f_{n+1}f_n - f_n^2 = (-1)^n.$$

Proof 10.

Let $P(n)$ be the statement that "$f_{n+1}^2 - f_{n+1}f_n - f_n^2 = (-1)^n$."

$\boxed{\text{Base Case}}$ Consider $P(1)$: $f_{1+1}^2 - f_{1+1}f_1 - f_1^2 = (-1)^1$. Since $f_1 = 1$ and $f_{1+1} = f_2 = 1$, we know that $f_{1+1}^2 - f_{1+1}f_1 - f_1^2 = 1^2 - (1)(1) - (1)^2 = 1 - 1 - 1 = -1$. Further since $(-1)^1 = -1$, so $f_{1+1}^2 - f_{1+1}f_1 - f_1^2 = (-1)^1$, and thus $P(1)$ is true.

$\boxed{\text{Inductive Steps}}$ Suppose $P(k)$ is true for some $k \in \mathbb{N}$. Then, $f_{k+1}^2 - f_{k+1}f_k - f_k^2 = (-1)^k$. Consider $P(k+1)$: $f_{k+1+1}^2 - f_{k+1+1}f_{k+1} - f_{k+1}^2 = f_{k+2}^2 - f_{k+2}f_{k+1} - f_{k+1}^2$. By definition of Fibonacci Sequence

(Definition 3.1), we know $f_{k+2} = f_k + f_{k+1}$. So,

$$\begin{aligned}
f_{k+2}^2 - f_{k+2}f_{k+1} - f_{k+1}^2 &= (f_k + f_{k+1})^2 - (f_k + f_k + 1)(f_{k+1}) - f_{k+1}^2 \\
&= f_k^2 + f_{k+1}^2 + 2f_k f_{k+1} - f_k f_{k+1} - f_{k+1}^2 - f_{k+1}^2 \\
&= f_k^2 + f_k f_{k+1} - f_{k+1}^2 \\
&= -(f_{k+1}^2 - f_{k+1}f_k - f_k^2) \\
&= -(-1)^k \\
&= (-1)^{k+1}.
\end{aligned}$$

Therefore, we get $P(k) \implies P(k+1)$.

Since we've proven $P(1)$ is true and $P(k) \implies P(k+1)$, by Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$.

■

3.9 Exam 2 Review 10

`Let` $f : \mathbb{N} \to \mathbb{N}$ `be defined recursively by` $f(1) = 1$ `and` $f(n+1) = \sqrt{2 + f(n)}$ `for all` $n \in \mathbb{N}$. `Prove that` $f(n) < 2$ `for all` $n \in \mathbb{N}$.

Proof 11.

Let $P(n)$ be the statement that "$f(n) < 2$, where f is a function from $\mathbb{N}$ to $\mathbb{N}$ defined recursively by $f(1) = 1$ and $f(n+1) = \sqrt{2 + f(n)}$."

$\boxed{\text{Base Case}}$ Consider $P(1)$. Note that, by definition of f, $f(1) = 1$ and $1 < 2$. So, $f(1) = 1 < 2$ and $P(1)$ is true.

$\boxed{\text{Inductive Steps}}$ Suppose $P(k)$ is true for some $k \geq 1$. That is, $f(k) < 2$. Consider $f(k+1) = \sqrt{2 + f(k)}$. Since $f(k) < 2$, we have $2 + f(k) < 2 + 2 = 4$. Hence, $f(k+1) = \sqrt{2 + f(k)} < \sqrt{4} = 2$. That is, $f(k+1) < 2$. So, $P(k) \implies P(k+1)$.

Since we've proven $P(1)$ is true and $P(k) \implies P(k+1)$, by mathematical induction, we know $P(n)$ is true for all $n \in \mathbb{N}$.

■

3.10 Exam 2 Review 11

`Prove that` $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$.

Proof 12.

Let $P(n)$ be $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$.

$\boxed{\text{Base Case}}$ Consider $P(1)$. Since $1^3 = 1$ and $\dfrac{1^2(1+1)^2}{4} = \dfrac{1^2(2)^2}{4} = \dfrac{4}{4} = 1$, so $1^3 = \dfrac{1^2(1+1)^2}{4}$. Hence, $P(1)$ is true.

$\boxed{\text{Inductive Steps}}$ Suppose $P(k)$ is true for some $k \geq 1$. Then,

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}. \qquad ①$$

Consider $P(k+1)$. Add $(k+1)^3$ to both sides of equation ①, we get

$$\begin{aligned}
1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{\left[k^2 + 4(k+1)\right](k+1)^2}{4} \\
&= \frac{(k+1)^2(k^2+4k+4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \frac{(k+1)^2[(k+1)+1]^2}{4}
\end{aligned}$$

Hence, $P(k) \implies P(k+1)$.

Since we've proven $P(1)$ is true and $P(k) \implies P(k+1)$, by Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$.

■

3.11 Exam 2 Review 18

`Let` $n \in \mathbb{Z}$ `and let` $S \subseteq \mathbb{Z}$ `satisfy` $|S| > n$. `Then, at least two distinct members of` S `are congruent` $\mod n$.

Proof 13.

WTS: $\exists\, a, b \in S$ *s.t.* $a \equiv b \mod n$, or $n \mid (a-b)$. $\forall s \in S$, we can write $s = nk + r$, where $k \in \mathbb{Z}$ and $r = \{0, 1, 2, \cdots, n\}$. There are exactly n possibilities for r; however, since $|s| > n$, there are more than n integers in S. So, by the Pigeonhole Principle, $\exists\, a, b \in S$ *s.t.* $a = nk + r$ and $b = nl + r$, where $k, l \in \mathbb{Z}$ and $r = \{0, 1, 2, \cdots, n\}$. So, $a - b = (nk + r) - (nl - r) = nk - nl = n(k - l)$. Since $k - l \in \mathbb{Z}$, we know $n \mid (a - b)$. So, $a \equiv b \mod n$.

■

4 Equivalence Relations

4.1 Exam 2 Review 6-b

Suppose that R is an equivalence relation on A and that $a, b \in A$. Then, if $[a] \cap [b] \neq \varnothing$, then $[a] = [b]$.

Proof 1.

Since $[a] \cap [b] \neq \varnothing$, $\exists x \in [a] \cap [b]$. By definition of set intersection, $x \in [a]$ and $x \in [b]$. Since $x \in [a]$, xRa. Also, since $x \in [b]$, then xRb. Since R is an equivalence relation, by symmetry, aRx. Since aRx and xRb, by transitivity, aRb. Then, $[a] = [b]$, by definition of equivalence class.

■

4.2 Exam 2 Review 12-a

Determine whether each of the following relations on $\mathbb{R}$ is an equivalence relation. Justify your answer. If R is an equivalence relation, describe its equivalence classes: xRy if $x - y \in \mathbb{Z}$.

Proof 2.

- **Reflexive**: Suppose $a \in \mathbb{R}$. Since $a - a = 0 \in \mathbb{Z}$, we have aRa. □
- **Symmetric**: Let $a, b \in \mathbb{R}$. Suppose aRb. Then, by definition, $a - b \in \mathbb{Z}$. That is, $\exists k \in \mathbb{Z}$ ***s.t.*** $a - b = k$. Consider $(b - a) = -(a - b) = -k$. Since $k \in \mathbb{Z}$, $-k \in \mathbb{Z}$. So, $b - a \in \mathbb{Z}$. That is, bRa. □
- **Transitive**: Let $a, b, c \in \mathbb{R}$. Suppose aRb and aRc. Then, by definition, $a - b \in \mathbb{Z}$ and $b - c \in \mathbb{Z}$. That is, $\exists\ k, l \in \mathbb{Z}$ ***s.t.*** $a - b = k$ and $b - c = l$. Add the two equations, we get $(a - b) + (b - c) = k + l$. Simplify, we will get $a - c = k + l$. Since $k, l \in \mathbb{Z}$, $k + l \in \mathbb{Z}$. So, $a - c \in \mathbb{Z}$, or aRc.

■

Claim. $[a] = \{a - k \mid k \in \mathbb{Z}\}$.

Proof 3.

($\subseteq$) Suppose $x \in [a]$. Then, by definition, aRx. So, $a - x \in \mathbb{Z}$. Suppose $a - x = m$ ***f.s.*** $m \in \mathbb{Z}$. Then, $-x = m - a$, or $x = a - m$. Since $m \in \mathbb{Z}$, $x \in \{a - k \mid k \in \mathbb{Z}\}$. □

($\supseteq$) Suppose $x \in \{a - k \mid k \in \mathbb{Z}\}$. Then, $x = a - m$ ***f.s.*** $m \in \mathbb{Z}$. Consider $a - x = a - (a - m) = a - a + m = m$. Since $m \in \mathbb{Z}$, $a - x \in \mathbb{Z}$. That is, aRx, or $x \in [a]$, by definition of equivalence class.

■

4.3 Exam 2 Review 12-b

`Determine whether each of the following relations on` $\mathbb{R}$ `is an equivalence relation. Justify your answer. If` R `is an equivalence relation, describe its equivalence classes:` xRy `if` $x + y \in \mathbb{Z}$.

Disproof 4.

R is not an equivalence relation because it is not reflexive. Suppose $a \in \mathbb{R}$. Then, $a + a = 2a \in \mathbb{R}$, but it does not always hold that $2a \in \mathbb{Z}$. Therefore, $a \not R a$, or R is not reflexive.

■

4.4 Exam 2 Review 13

`Prove or disprove:` R `is an equivalence relation on` $\mathbb{Z}$. `If` R `is an equivalence relation, describe its equivalence classes:` xRy `if` $4 \mid (x + y)$.

Disproof 5.

R is not an equivalence relation because it is not reflexive. Suppose $a \in \mathbb{Z}$. Consider $a + a = 2a$. Since $a \in \mathbb{Z}$, $2a \in \mathbb{Z}$, but $4 \nmid 2a$ for all $a \in \mathbb{Z}$. Therefore, $a \not R a$, and so R is not reflexive.

■

4.5 Exam 2 Review 14

`Prove or disprove:` R `is an equivalence relation on` $\mathbb{Z}$. `If` R `is an equivalence relation, describe its equivalence classes:` xRy `if` $4 \mid (x + 3y)$.

Proof 6.

- **Reflexive**: Suppose $a \in \mathbb{Z}$. Consider $a + 3a = 4a$. Since $a \in \mathbb{Z}$, $4 \mid 4a$. That is, $4 \mid a + 3a$, or aRa. □

- **Symmetric**: Suppose $a, b \in \mathbb{Z}$. Then, $a + 3b = 4k$ *f.s.* $k \in \mathbb{Z}$. So, $a = 4k - 3b$. Consider

$$\begin{aligned} b + 3a = b + 3(4k - 3b) &= b + 12k - 9b \\ &= 12k - 8b \\ &= 4(3k - 2b). \end{aligned}$$

Since $k, b \in \mathbb{Z}$, $4k - 2b \in \mathbb{Z}$. So, $4 \mid 4(3k - 2b)$, or $4 \mid b + 3a$. Hence, bRa. □

- **Transitive**: Let $a, b, c \in \mathbb{Z}$. Suppose aRb and bRc. Then, $4 \mid a + 3b$ and $4 \mid b + 3c$. Hence, $\exists\, k, l \in \mathbb{Z}$ *s.t.* $a + 3b = 4k$ and $b + 3c = 4l$. Hence, $a = 4k - 3b$ and $3c = 4l - b$. Consider

$$a + 3c = 4k - 3b + 4l - b = 4k + 4l - 4b = 4(k + l - b).$$

Since $k, b, l \in \mathbb{Z}$, $k + l - b \in \mathbb{Z}$. So, $4 \mid 4(k + l - b)$, or $4 \mid a + 3c$. Therefore, aRc. $\square$

Since R is symmetric, reflexive, and transitive, R is an equivalence relation.

$\blacksquare$

Claim. $[i] = \{4k + i \mid k \in \mathbb{Z}\} \quad \forall i \in \{0, 1, 2, 3\}$.

Proof 7.

($\subseteq$) Suppose $x \in [i]$. Then, xRi. By definition of R, $4 \mid x + 3i$. So, $x + 3i = 4k$ *f.s.* $k \in \mathbb{Z}$. Then,

$$x = 4k - 3i = 4(k - i) - 3i + 4i = 4(k - i) + i.$$

Since $k \in \mathbb{Z}$ and $i \in \{0, 1, 2, 3\}$, we know $k - i \in \mathbb{Z}$. Then, $x = 4(k - i) + i \in \{4k + i \mid k \in \mathbb{Z}\}$. $\square$

($\supseteq$) Suppose $x \in \{4k + i \mid k \in \mathbb{Z}\}$. Then, $x = 4k + i$ *f.s.* $k \in \mathbb{Z}$. Consider

$$x + 3i = 4k + i + 3i = 4k + 4i = 4(k + i).$$

Since $k \in \mathbb{Z}$ and $i \in \{0, 1, 2, 3\}$, we know $k + i \in \mathbb{Z}$. Then, $4 \mid 4(k + i)$, or $x \mid x + 3i$. That is, xRi, or $x \in [i]$.

$\blacksquare$

4.6 Exam 2 Review 15

`Define a relation` R `on` $\mathbb{R}^2$ `as follows: for all` $(a_1, b_1), (a_2, b_2) \in \mathbb{R}^2, (a_1, b_1)R(a_2, b_2)$ `if` (a_1, b_1) `and` (a_2, b_2) `are on the same line through the origin. Decide whether` R `is an equivalence relation - either show why or why not. If it is, what are the elements of the equivalence class` $[(1, 2)]$`?`

Proof 8.

- **Reflexive**: Suppose $(a, b) \in \mathbb{R}^2$. The line of (a, b) and the origin is $y = \frac{b}{a}x$. Apparently, (a, b) and (a, b) is both on $y = \frac{b}{a}x$. So, $(a, b)R(a, b)$. $\square$

- **Symmetric**: Suppose (a_1, b_1) and $(a_2, b_2) \in \mathbb{R}^2$. Let $(a_1, b_1)R(a_2, b_2)$. The line between (a_1, b_1) and the origin is $y = \frac{b_1}{a_1}x$. Then, (a_2, b_2) is on the same line: $b_2 = \frac{b_1}{a_1} \cdot a_2$. So, $\frac{b_2}{a_2} = \frac{b_1}{a_1}$. That is,

$\dfrac{b_2}{a_2} \cdot a_1 = b_1$, or (a_1, b_1) is on the line $y = \dfrac{b_2}{a_2}x$. Since $y = \dfrac{b_2}{a_2}x$ is the line between (a_2, b_2) and $(0, 0)$, we have $(a_2, b_2)R(a_1, b_1)$. $\square$

- **Transitive**: Suppose $(a_1, b_1), (a_2, b_2)$, and $(a_3, b_3) \in \mathbb{R}^2$. Suppose $(a_1, b_1)R(a_2, b_2)$ and $(a_2, b_2)R(a_3, b_3)$. Then, $\dfrac{b_1}{a_1} = \dfrac{b_2}{a_2}$ and $\dfrac{b_2}{a_2} = \dfrac{b_3}{a_3}$. So, $\dfrac{b_1}{a_1} = \dfrac{b_2}{a_2} = \dfrac{b_3}{a_3}$. Then, $(a_1, b_1)R(a_3, b_3)$.

$\blacksquare$

Claim. $[(1, 2)] = \{(x, y) \mid y = 2x\}$.

Proof 9.

($\subseteq$) Suppose $(x, y) \in [(1, 2)]$. Then, $(x, y)R(1, 2)$. So, $\dfrac{y}{x} = \dfrac{2}{1}$. That is, $y = 2x$. Hence, $(x, y) \in \{(x, y) \mid y = 2x;\ x, y \in \mathbb{R}\}$. $\square$

($\supseteq$) Suppose $(x, y) \in \{(x, y) \mid y = 2x\}$. Then, $(x, y) = (x, 2x)$. Since $\dfrac{2x}{x} = \dfrac{2}{1}$, we have $(x, 2x)R(1, 2)$. Therefore, $(x, y) \in [(1, 2)]$.

$\blacksquare$

5 Functions

> *5.1 Exam 2 Review 17*
>
> `Let` $A = \{x, y, z\}$. `Define functions` $f : \mathcal{P}(A)$ `by` $f(a) = \{a\}$ `and` $g : A \to \mathcal{P}(A)$ `by` $g(a) = A - \{a\}$. `Find` $\mathrm{Im}(f)$ `and` $\mathrm{Im}(g)$.

Claim. $\mathrm{Im}(f) = \{\{x\}, \{y\}, \{z\}\}$.

Proof 1.

($\subseteq$) Suppose $a \in A$. Then, we have $f(a) = \{a\}$. Since $a \in A$, $\{a\} \in \{\{x\}, \{y\}, \{z\}\}$. Therefore, $\mathrm{Im}(f) \subseteq \{\{x\}, \{y\}, \{z\}\}$. $\square$

($\supseteq$) Suppose $a \in \{\{x\}, \{y\}, \{z\}\}$. WLOG, suppose $a = \{x\}$. Choose $b = x$. So, $f(b) = \{b\} = \{x\} = a$. Therefore, $a \in \mathrm{Im}(f)$. That is, $\{\{x\}, \{y\}, \{z\}\} \subseteq \mathrm{Im}(f)$.

$\blacksquare$

Claim. $\mathrm{Im}(g) = \{\{y, z\}, \{x, z\}, \{x, y\}\}$.

Proof 2.

($\subseteq$) Suppose $a \in A$. Then, $a = x$, or $a = y$, or $a = z$. WLOG, suppose $a = x$. Then,

$$f(a) = A - \{a\} = \{x, y, z\} - \{x\} = \{y, z\}.$$

Since $\{y, z\} \subseteq \{\{y, z\}, \{x, z\}, \{x, y\}\}$, we know that $f(a) \in \{\{y, z\}, \{x, z\}, \{x, y\}\}$. Therefore, we've proven $\mathrm{Im}(f) \subseteq \{\{y, z\}, \{x, z\}, \{x, y\}\}$. $\square$

($\supseteq$) Suppose $a \in \{\{y, z\}, \{x, z\}, \{x, y\}\}$. WLOG, suppose $a = \{y, z\}$. Note that

$$\exists x \in A \text{ s.t. } f(x) = A - \{x\} = \{y, z\} = a.$$

So, $a \in \mathrm{Im}(f)$. That is, $\{\{y, z\}, \{x, z\}, \{x, y\}\} \subseteq \mathrm{Im}(f)$.

$\blacksquare$

> *5.2 Exam 2 Review 17*
>
> `Let` $f : \mathbb{R} \to \mathbb{R}$ `be given by` $f(x) = 2x^3 + 3x^2 - 12x + 1$. `Let` $X = [-1, 2]$. `Find` $f(X)$.

Answer 3.

Find $f'(x) = 6x^2 + 6x - 12$. So, $f(x)$ is not always increasing or decreasing. Find critical points by setting $f'(x) = 0$: $6x^2 + 6x - 12 = 0$, so we get $(x + 2)(x - 1) = 0$, or $x = -2,\ x = 1$. Since $X = [-1, 2]$, it must be $x = 1$. Check $f''(x) = 12x + 6 :\ f''(1) = 12(1) + 6 = 12 + 6 > 0$. So, $f(1)$ is the minimum value: $f(1) = 2(1)^3 + 3(1)^2 - 12(1) + 1 = -6$. Then, maximum value will be found at $x = -1$ or $x = 2$. At $x = -1$, $f(-1) = 2(-1)^3 + 3(-1)^2 - 12(-1) + 1 = 14$. At $x = 2$, we have $f(2) = 2(2)^3 + 3(2)^2 - 12(2) + 1 = 5$. Since $14 > 5$, maximum value occurs at $x = -1$. So, $f(X) = [-6, 14]$.

□

Definition 5.1 ($\varepsilon - \delta$ Definition of Continuity) *Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x)$, then f is continuous at $x = a$ when then following condition is satisfied:*

$$\forall \varepsilon > 0, \exists \delta \in \mathbb{R} \text{ s.t. } |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$$

5.3 Exam 3 Review 2

`Consider the function` $f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$. `Rigorously prove that` f `is discontinuous at` $x = 0$. `Your proof should involve` ε `and` δ.

Proof 4.

Choose $\varepsilon = \frac{1}{2}$. Then, we need $|f(x) - f(0)| < \frac{1}{2}$. That is, we want $|f(x) - 1| < \frac{1}{2}$, or $-\frac{1}{2} < f(x) - 1 < \frac{1}{2}$. That is, $\frac{1}{2} < f(x) < \frac{3}{2}$. Note that $\forall x \in (-\delta, 0)$, $f(x) = 0$, by definition of (x). That is, $f(x) \notin \left(\frac{1}{2}, \frac{3}{2}\right)$. So, f is discontinuous at $x = 0$.

■

5.4 Exam 3 Review 3-a

`Use the formal definition of continuity, prove that the function` $f(x) = x^2 + 4x + 3$ `is continuous at` $x = -2$.

Proof 5.

Let $\varepsilon > 0$ be given. Suppose $\delta = \sqrt{\varepsilon}$. Since $\varepsilon > 0$, we know $\delta = \sqrt{\varepsilon} > 0$. Suppose $|x - (-2)| = |x + 2| < \delta$. Then,

$$\begin{aligned} |f(x) - f(-2)| = \left|x^2 + 4x + 3 - (-1)\right| = \left|x^2 + 4x + 3 + 1\right| &= \left|x^2 + 4x + 4\right| \\ &= \left|(x + 2)^2\right| \\ &= |x + 2||x + 2| \\ &< \delta \cdot \delta = \sqrt{\varepsilon} \cdot \sqrt{\varepsilon} = \varepsilon. \end{aligned}$$

Since ε was arbitrary, we've shown that

$$\forall \varepsilon > 0, \exists\, \delta = \sqrt{\varepsilon} > 0 \; s.t. \; |x+2| > \delta \implies |f(x) - f(-2)| < \varepsilon.$$

So, f is continuous at $x = -2$.

■

> *5.5 Exam 3 Review 3-b*
>
> `Use the formal definition of continuity, prove that the function` $f(x) = x^2 + 4x + 3$ `is continuous at` $x = 2$.

Proof 6.

Let $\varepsilon > 0$ be given. Suppose $\delta = \min\left\{1, \dfrac{\varepsilon}{9}\right\}$. Then, $\delta \leq 1$ and $\delta \leq \dfrac{\varepsilon}{9}$. Suppose $x \in \mathbb{R}$ and $|x-2| < \delta$. Since $|x-2| < \delta \leq 1$, we have $1 < x < 3$. So, $7 < x + 6 < 9$. That is, $|x+6| < 9$. Then,

$$\begin{aligned}
|f(x) - f(2)| = \left|x^2 + 4x + 3 - 15\right| &= \left|x^2 + 4x - 12\right| \\
&= |(x-2)(x+6)| \\
&= |x-2||x+6| \\
&< 9|x-2| \\
&< 9 \cdot \delta \\
&\leq 9 \cdot \frac{\varepsilon}{9} = \varepsilon.
\end{aligned}$$

Since ε was arbitrary, we've proven that

$$\forall \varepsilon > 0, \exists\, \delta = \min\left\{1, \frac{\varepsilon}{9}\right\} > 0 \; s.t. \; |x-2| < \delta \implies |f(2) - f(x)| < \varepsilon.$$

So, by the definition of continuity, f is continuous at $x = 2$.

■

> *5.6 Exam 3 Review 6*
>
> `Prove or disprove: Every injective map form` $\mathbb{R} \to \mathbb{R}$ `is bijective.`

Disproof 7.

Consider $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = e^x$. For $x, y \in \mathbb{R}$, if $f(x) = f(y)$, we have $e^x = e^y$. Take logarithm with base e, we have $\ln e^x = \ln e^y$. So, $x = y$. Hence, f is injective. Consider $b = -1 \in \mathbb{R}$. Set

$f(x) = -1$. That is, $e^x = -1$. ⋇ This contradicts with the fact that $f(x) > 0$. Therefore, our assumption is wrong, and $f(x)$ cannot be -1. Hence, by definition, f is not surjective.

■

5.7 Exam 3 Review 7

Show that the function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ defined by $f(x) = \dfrac{x+1}{x}$ is injective but not surjective. How could we change the codomain so that f is surjective?

Proof 8.

- **Injective**: Suppose $x, y \in \mathbb{R} - \{0\}$ *s.t.* $f(x) = f(y)$. Then, we get

$$\frac{x+1}{x} = \frac{y+1}{y}$$
$$(x+1)y = (y+1)x$$
$$xy + y = xy + x$$
$$y = x.$$

  So, $f(x) = f(y) \implies x = y$. That is, f is injective. □

- **Not Surjective**: Set $f(x) = 1$. So we should have $\dfrac{x+1}{x} = 1$. So, $x + 1 = x$, or $1 = 0$. This is not possible, so $f(x) \neq 1$. Therefore, f is not surjective.

■

Answer 9.

We can change the codomain to $\mathbb{R} - \{1\}$. So that our function will become surjective.

□

5.8 Exam 3 Review 11-a

Let $f : A \to B$ for a function and $X \subseteq A$. Prove or disprove: $f^{-1}(f(X)) = X$.

Disproof 10.

Consider $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Define $f(1) = a$ and $f(2) = f(3) = b$. Set $X = \{2\}$, then $f(X) = f(\{2\}) = \{b\}$. Therefore, $f^{-1}(f(X)) = f^{-1}(\{b\}) = \{2, 3\}$. Since $3 \in f^{-1}(f(X))$ but $3 \notin X$, $f^{-1}(f(X)) \neq X$.

■

5.9 Exam 3 Review 11-b

`Let` $f : A \to B$ `for a function and` $X \subseteq A$`. Prove or disprove:` $f(f^{-1}(f(X))) = f(X)$.

Proof 11.

Let $f : A \to B$ be a function and $X \subseteq A$.

($\subseteq$) Suppose $x \in f(f^{-1}(f(X)))$. Then, $\exists a \in f^{-1}(f(X))$ *s.t.* $f(a) = x$. Since $a \in f^{-1}(f(X))$, $f(a) \in f(X)$. Note $f(a) = x$, so $x \in f(X)$. $\square$

($\supseteq$) Suppose $x \in f(X)$. Then, $\exists a \in X$ *s.t.* $f(a) = x$. Since $f(a) = x \in f(X)$, we have $f(a) \in f(X)$. Then, $a \in f^{-1}(f(X))$. Therefore, $f(a) \in f(f^{-1}(f(X)))$. That is, $x \in f(f^{-1}(f(X)))$.

$\blacksquare$

5.10 Exam 3 Review 12

`Let` $f : A \to B$ `and` $g : B \to C$`, and assume that` f `is surjective. Prove that` $g \circ f$ `is injective if and only if` g `and` f `are both injective.`

Proof 12.

($\Rightarrow$) Suppose $g \circ f$ is injective.

- f **injective**: Let $x, y \in A$ *s.t.* $f(x) = f(y)$. Apply g on both sides, we get $g(f(x)) = g(f(y))$. That is, $(g \circ f)(x) = (g \circ f)(y)$. Since $(g \circ f)$ is injective, we have $x = y$. Hence, f is injective.

- g **injective**: Let $x, y \in B$ *s.t.* $g(x) = g(y)$. Since f is surjective from $A \to B$, $\exists\, a, b \in A$ *s.t.* $f(a) = x$ and $f(b) = y$. Then, $g(x) = g(f(a)) = (g \circ f)(a)$ and $g(y) = g(f(b)) = (g \circ f)(b)$. Therefore, $(g \circ f)(a) = (g \circ f)(b)$. Since $g \circ f$ is injective, we have $a = b$. Since f is injective (proven above), we have $f(a) = f(b)$. Since $f(a) = x$ and $f(b) = y$, we know $x = y$, and hence g is also injective.
  $\square$

($\Leftarrow$) Suppose g and f are injective. Let $x, y \in A$ *s.t.* $(g \circ f)(x) = (g \circ f)(y)$. Since $(g \circ f)(x) = g(f(x))$ and $(g \circ f)(y) = g(f(y))$, we have $g(f(x)) = g(f(y))$. Since g is injective, we know $f(x) = f(y)$. Further since f is also injective, we have $x = y$. Then, $g \circ f$ is injective.

$\blacksquare$

5.11 Exam 3 Review 13

`Suppose that` $f : A \to B$ `is a function. Prove that` f `is injective if and only if for all subsets` C, D `of` A`,` $f(C \cap D) = f(C) \cap f(D)$.

Proof 13.

Let $f : A \to B$ be an injective function. Let $C, D \subseteq A$.

($\Rightarrow$)Suppose f is injective. WTS: $f(C \cap D) = f(C) \cap f(D)$.

($\subseteq$) Let $x \in f(C \cap D)$. Then, $\exists\, a \in C \cap D$ *s.t.* $f(a) = x$. Since $a \in C \cap D$, we have $a \in C$ and $a \in D$. Since $a \in C$, $f(a) \in f(C)$ That is, $x \in f(C)$. Similarly, since $a \in D$, $f(a) \in f(D)$, and thus $x \in f(D)$. Since $x \in f(C)$ and $x \in f(D)$, by definition of set intersection, $x \in f(C) \cap f(D)$. $\square$

($\supseteq$) Let $x \in f(C) \cap f(D)$. Then, $x \in f(C)$ and $x \in f(D)$. So, $\exists\, c \in C$ *s.t.* $f(c) = x$ and $\exists\, d \in D$ *s.t.* $f(d) = x$. Therefore, we know $f(c) = f(d) = x$. Since f is injective, we have $c = d$. Hence, $c \in C$ and $c \in D$, and that is, $c \in C \cap D$. So, $f(c) = x \in f(C \cap D)$. $\square$

($\Leftarrow$) Suppose $f(C \cap D) = f(C) \cap f(D)$. Suppose $x, y \in A$ *s.t.* $f(x) = f(y)$. Say $f(x) = f(y) = m$. Suppose $C = \{x\} \subseteq A$ and $D = \{y\} \subseteq A$. Then, by assumption, $f(C) = f(\{x\}) = \{m\}$ and $f(D) = f(\{y\}) = \{m\}$. So, $f(C \cap D) = f(C) \cap f(D) = \{m\} \cap \{m\} = \{m\}$. If $C \cap D = \varnothing$, then $f(C \cap D) = f(\varnothing) = \varnothing \neq \{m\}$. Hence, $C \cap D \neq \varnothing$. That is, $\{x\} \cap \{y\} \neq \varnothing$. The only way for intersection of two single-element sets being non-empty is that the two elements are identical. So, $x = y$.

$\blacksquare$

5.12 Exam 3 Review 14

`Let` A, B `be sets, and let` $F(A, B)$ `denote the set of all functions from` A `to` B`. Let` $g : A \to A$ `be a bijection. Define a new function` $\Delta_g : F(A, B) \to F(A, B)$ `as follows:` $f \mapsto f \circ g$`. Prove that` Δ_g `is a bijection.`

Proof 14.

- **Injective**: Suppose $f, h \in F(A, B)$ *s.t.* $\Delta_g(f) = \Delta_g(h)$. Since g is a bijection, g is also invertible. Denote the inverse of g as g^{-1}. By definition, $\Delta_g(f) = f \circ g$ and $\Delta_g(h) = h \circ g$. So, by assumption, $f \circ g = h \circ g$. Apply $f \circ g$ and $h \circ g$ to g^{-1}, respectively, we have $(f \circ g) \circ g^{-1} = (h \circ g) \circ g^{-1}$. So, we know that $f \circ (g \circ g^{-1}) = h \circ (g \circ g^{-1})$. Since $g \circ g^{-1} = i_A$, we have $f \circ i_A = h \circ i_A$. That is, $f = h$. $\square$

- **Surjective**: Suppose $h \in F(A, B)$. Choose $f = h \circ g^{-1} \in F(A, B)$. Then,

$$\Delta_g(f) = f \circ g = (h \circ g^{-1}) \circ g = h \circ (g^{-1} \circ g) = h \circ i_A = h.$$

Therefore, $\exists f = h \circ g^{-1} \in F(A, B)$ *s.t.* $\Delta_g(f) = h \quad \forall h \in F(A, B)$. That is, Δ_g is surjective.

$\blacksquare$

5.13 Exam 3 Review 15-a

`Let` A, B `be sets, and let` $f : A \to B$ `be a function. Let` I `be an index, and let` $\{C_i\}_{i\in I}$ `be a collection of subsets such that for all` $i \in I, C_i \subseteq B$. `Prove that` $f^{-1}\left(\bigcap_{i\in I} C_i\right) = \bigcap_{i\in I} f^{-1}(C_i)$.

Proof 15.

($\subseteq$) Suppose $a \in f^{-1}\left(\bigcap_{i\in I} C_i\right)$. So, by definition of inverse image, $f(a) \in \bigcap_{i\in I} C_i$. That is, $\forall i \in I,\ f(a) \in C_i$. By definition of inverse image, $a \in f^{-1}(C_i) \quad \forall i \in I$. That is, $a \in \bigcap_{i\in I} f^{-1}(C_i)$, by definition of set intersection. $\square$

($\supseteq$) Suppose $a \in \bigcap_{i\in I} f^{-1}(C_i)$. By definition of set intersection, $a \in f^{-1}(C_i) \quad \forall i \in I$. By definition of inverse image, $f(a) \in C_i \quad \forall i \in I$. That is, $f(a) \in \bigcap_{i\in I} C_i$. So, $a \in f^{-1}\left(\bigcap_{i\in I} C_i\right)$.

$\blacksquare$

5.14 Exam 3 Review 15-a

`Let` A, B `be sets, and let` $f : A \to B$ `be a function. Let` I `be an index, and let` $\{C_i\}_{i\in I}$ `be a collection of subsets such that for all` $i \in I, C_i \subseteq B$. `Prove that` $f^{-1}\left(\bigcup_{i\in I} C_i\right) = \bigcup_{i\in I} f^{-1}(C_i)$.

Proof 16.

($\subseteq$) Suppose $a \in f^{-1}\left(\bigcup_{i\in I} C_i\right)$. By definition of inverse image, $f(a) \in \bigcup_{i\in I} C_i$. Hence, by definition of set union, $f(a) \in C_k$ *f.s.* $k \in I$. So, $a \in f^{-1}(C_k)$ *f.s.* $k \in I$. Since $f^{-1}(C_k) \subseteq \bigcup_{i\in I} f^{-1}(C_i)$, we have $a \in \bigcup_{i\in I} f^{-1}(C_i)$. $\square$

($\supseteq$) Suppose $a \in \bigcup_{i\in I} f^{-1}(C_i)$. Then, by definition of set union, $a \in f^{-1}(C_k)$ *f.s.* $k \in I$. By definition of inverse image, $f(a) \in C_k$ *f.s.* $k \in I$. So, $f(a) \in \bigcup_{i\in I} C_i$. That is, $a \in f^{-1}\left(\bigcup_{i\in I} C_i\right)$.

$\blacksquare$