Emory University

MATH 351 Partial Differential Equations

Learning Notes

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Contents

1 Introduction

1.1 Ways to Study Differential Equations

- Qualitative: analyze the behavior of the solution
- Quantitative: find the solution
- Approximation: numerical solvers.

1.2 Review and Preview

- $y = y(t)$ is a solution to an ODE.
    1. $y(t)$ is a one-variable function.
    2. $y(t, C) = y(t) + C$ defines a family of solutions, where C is a constant.
    3. Order of ODE: highest order of derivative.
- Similar definitions apply to a PDE: $u = u(t, x)$, where $u(t, x)$ is a function of two or more variables.
- Famous PDEs:
    1. Heat equation in 1D:
    $$u_t = u_{xx}$$
    2. Heat equation in 2D:
    $$u_t = u_{xx} + u_{yy}$$
    3. Laplace equation in polar coordinate:
    $$u_{tt} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$
    4. Wave equation in 3D:
    $$u_{tt} = u_{xx} + u_{yy} + u_{zz}$$

1.3 Classification of PDEs and Definitions

- Order: Highest derivative of the PDE.
- Linearity: The PDE can be written as

$$Lu = f,$$

where L is some linear operator.

- Homogeneity (only for linear PDEs): $f = 0$ means homogenous; $f \neq 0$ means non-homogenous.
- Number of variables: 2 or more.
- Kinds of Coefficients: Constant/Non-constant.

Definition 1.3.1 (Linear PDEs with 2 Variables). We can write

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \tag{1}$$

where A, B, C, D, E, F, G are independent of u (but they can be dependent on x or y).

- If $G = 0$, (1) is *homogenous*. If $G \neq 0$, (1) is *non-homogenous*.
- Similar to determinants for quadratic equations, we also classify PDEs according to the sign of $B^2 - 4AC$:
  1. *Parabolic* if $B^2 - 4AC = 0$,
  2. *Hyperbolic* if $B^2 - 4AC > 0$, and
  3. *Elliptic* if $B^2 - 4AC < 0$.

Definition 1.3.2 (Initial Value/Boundary Value Problems).

- If initial conditions are provided, we have an *Initial Value Problem (IVP)*.
- If boundary conditions are given, we have a *Boundary Value Problem (BVP)*.
- If both initial conditions and boundary conditions are provided, we have an *Initial Boundary Value Problem (IBVP)*.

Example 1.3.3 Example of an IBVP Problem

$$\begin{cases} \text{[PDE]} & \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0, \quad \text{with } 0 < x < 1, t > 0, \\ \text{[BCs]} & \begin{cases} u(t,0) = 0 \\ u(t,1) = e^{-t}, \end{cases} \quad \text{with } t > 0, \\ \text{[IC]} & u(0,x) = x, \quad \text{with } 0 < x < 1. \end{cases} \tag{2}$$

With an IBVP, we can

- Solve this IBVP, or
- Verify a function is a solution for the IBVP.

Example 1.3.4 Classification of PDEs

- $u_t t = e^{-t} u_x x + \sin t$

  2^{nd} order, 2 variables (t, x), linear, non-homogenous.

- $u u_{xx} + u_t = 0$

  2^{nd} order, 2 variables (t, x), nonlinear.

- $u_{xx} + y u_{yy} = 0$

  2^{nd} order, 2 variables (x, y), linear, homogenous.

- $x u_x + y u_y + u^2 = 0$

  1^{st} order, 2 variables (x, y), nonlinear.

Classify the following second order PDE as parabolic, hyperbolic, or elliptic. (*Only second order linear PDE can be classified.*)

- $u_t = u_{xx} \implies u_{xx} - u_t = 0$.

  $A = 1, B = 0, C = 0 \implies B^2 - 4AC = 0 \implies$ parabolic.

- $u_{tt} = u_{xx} \implies u_{xx} - u_{tt} = 0$.

  $A = 1, B = 0, C = -1 \implies B^2 - 4AC = 4 > 0 \implies$ hyperbolic.

- $u_{tx} = 0$.

  $A = 0, B = 1, C = 0 \implies B^2 - 4AC = 1 > 0 \implies$ hyperbolic.

- $u_{xx} + u_{yy} = 0$.

  $A = 1, B = 0, C = 1 \implies B^2 - 4AC = -4 < 0 \implies$ elliptic.

- $y u_{xx} + u_{yy} = 0$.

  $$A = y, B = 0, C = 1 \implies B^2 - 4AC = -4y \implies \begin{cases} \text{elliptic if } y > 0 \\ \text{parabolic if } y = 0 \\ \text{hyperbolic if } y < 0 \end{cases}.$$

Classify the following PDE. Classify them as parabolic, hyperbolic, or elliptic when applicable.

- $u_t = u_{xx} + 2u_x + u \implies u_{xx} + 2u_x - u_t + u = 0$.

  2^{nd} order, 2 variables (t, x), linear, homogenous.

  $A = 1, B = 0, C = 0 \implies B^2 - 4AC = 0 \implies$ parabolic.

- $u_t = u_{xx} + e^{-t} \implies u_{xx} - u_t + e^{-t} = 0$.

  2^{nd} order, 2 variables (t, x), linear, non-homogenous.

  $A = 1, B = 0, C = 0 \implies B^2 - 4AC = 0 \implies$ parabolic.

- $u_{xx} + 3u_{xy} + u_{yy} = \sin x$

  2^{nd} order, 2 variables (x, y), linear, non-homogenous.

  $A = 1, B = 3, C = 1 \implies B^2 - 4AC = 9 - 4 > 0 \implies$ hyperbolic.

- $u_{tt} = uu_{xxxx} + e^{-t}$

  4^{th} order, 2 variables (t, x), nonlinear, non-homogenous.

2 First Order Linear PDEs

2.1 Principle of Superposition

Theorem 2.1.1 Principle of Superposition

If $u_1(x,y)$ and $u_2(x,y)$ are solutions of

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \tag{3}$$

then $u_1(x,y) + u_2(x,y)$ is also a solution of the equation if and only if (3) is homogenous.

Proof 1. Let's plug-in $u_1(x,y) + u_2(x,y)$ to the LHS:

$$\begin{aligned}
&A(u_1+u_2)_{xx} + B(u_1+u_2)_{xy} + C(u_1+u_2)_{yy} + D(u_1+u_2)_x + E(u_1+u_2)_y + F(u_1+u_2) \\
=&Au_{1xx} + Bu_{1xy} + Cu_{1yy} + Du_{1x} + Eu_{1y} + Fu_1 + Au_{1xx} + Bu_{2xy} + Cu_{2yy} + Du_{2x} + Eu_{2y} + Fu_2 \\
=&G + G \\
=&2G.
\end{aligned}$$

So, $u_1 + u_2$ is a solution $\iff G = 2G \iff G = 0 \iff$ (3) is homogenous. ■

Example 2.1.2 Simple PDEs

Solve the following PDEs:

- $\dfrac{\partial u(x,y)}{\partial x} = 0.$

  Solution 2.

  $u(x,y) = f(y)$, only a function of y. □

- $\dfrac{\partial^2 u(x,y)}{\partial x \partial y} = 0.$

  Solution 3.

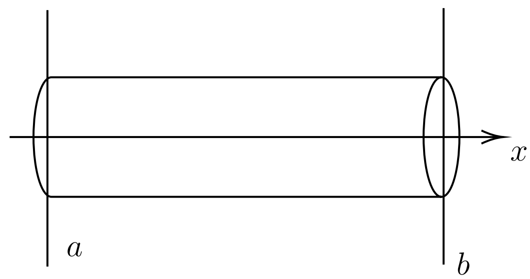
  There are two possible orders to take the derivatives:

  1. $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) = 0 \implies \dfrac{\partial u}{\partial y}$ is a function of y.
  2. $\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right) = 0 \implies \dfrac{\partial u}{\partial x}$ is a function of x.

  Combing the two cases, we know: $u(x,y) = f(x) + g(y)$. □

2.2 Transport Equation and Method of Characteristics

Consider we have a bloodstream, and we want to model the oxygen level in it. Let $u(t, x)$ be the concentration or density of oxygen, and $q(t, x, u)$ be the flux.



Then, the total mass of oxygen at time t is given by

$$\Theta(t) = \int_a^b u(t, x)\,\mathrm{d}x.$$

By the conservation law, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}[\Theta(t)] &= q(t, a, u) - q(t, b, u) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_a^b u(t, x)\,\mathrm{d}x\right] &= q(t, a, u) - q(t, b, u) \\
&= -\Big[q(t, b, u) - q(t, a, u)\Big] \\
&= -\int_a^b \frac{\partial q}{\partial x}\,\mathrm{d}x && \text{[Fund. Thm. of Calculus]} \\
\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(t, x)\,\mathrm{d}x &= -\int_a^b \frac{\partial q}{\partial x}\,\mathrm{d}x \\
\int_a^b \frac{\partial u}{\partial t}\,dx &= -\int_a^b \frac{\partial q}{\partial x}\,\mathrm{d}x && \text{[Interchange derivative and integral]} \\
\int_a^b \frac{\partial u}{\partial t} + \frac{\partial q}{\partial x}\,\mathrm{d}x &= 0 \\
\implies & \boxed{\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0} && \text{(Transport Equation)}
\end{aligned}$$

Now, assume q is linear in u. Then,

$$q(t, x, u) = c \cdot u(t, x),$$

where c is the velocity which does not depend on u. Then, (Transport Equation) becomes

$$\frac{\partial u}{\partial t} + c \cdot \frac{\partial u}{\partial x} = 0 \quad \text{with IC } u(0, x) = u_0(x).$$

The classification is: 1^{st} order, 2 variables (t, x), linear, and homogenous PDE.

To solve it, suppose we are moving on the stream, at the same velocity as the stream, to make observations. Let $x(t)$ denote our trajectory. Let $x(t)$ be a function of t. Then,

$$u(t, x) = u(t, x(t)).$$

Then, the *total derivative* of u becomes

$$\frac{D}{Dt}u(t, x(t)) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t},$$

where $\frac{\mathrm{d}x}{\mathrm{d}t} = c$ is exactly the velocity in (Transport Equation). We form two differential equations to solve:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = c \\ x(0) = x_0 \end{cases} \quad \text{and} \quad \begin{cases} \frac{D}{Dt}u(t, x(t)) = 0 \\ u(0, x(0) = u_0(x_0). \end{cases}$$

From the first system, we get $x(t) = ct + x_0$ ①. From the second system we know that $u(t, x(t))$ is a constant since the total derivative is 0. By the IC, we know $u(t, x(t)) = \text{IC} = u_0(x_0)$ ②.

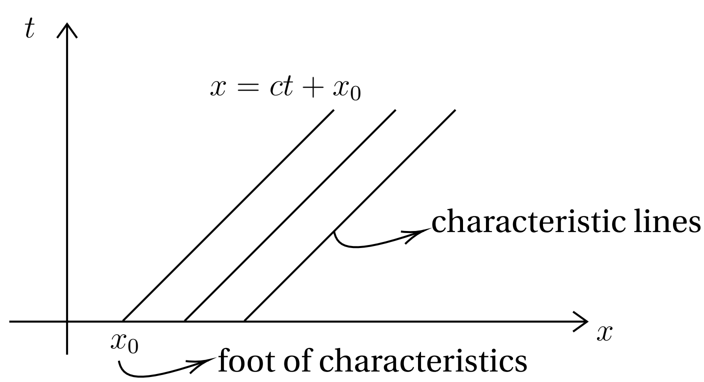
Now, jumping out from the stream, we want everything in (t, x). From ①:

$$x_0(t, x) = x - ct.$$

Substitute this into ②, we get

$$u(t, x) = u_0(x - ct).$$

This method of solving the PDE is called the *Method of Characteristics*. Graphically,



Example 2.2.1

$$\frac{\partial u}{\partial t} + 5\frac{\partial u}{\partial x} = 0 \quad \text{with } u(0, x) = e^x \sin x.$$

Solution 1.

$$\frac{D}{Dt}u(t,x(t)) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t}.$$

So, we have two ODEs:

$$① \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = 5 \\ x(0) = x_0 \end{cases} \qquad ② \begin{cases} \dfrac{Du}{Dt} = 0 \\ u(0,x(0)) = e^{x_0}\sin x_0. \end{cases}$$

From ①, we have

$$x = 5t + x_0$$
$$x_0 = x - 5t$$

From ②, since $\dfrac{Du}{Dt} = 0$, u is a constant. By the initial condition,

$$u(t,x(t)) = e^{x_0}\sin x_0$$
$$u(t,x) = e^{(x-5t)}\sin(x-5t).$$

□

2.2.1 What Happens when Velocity is not a Constant?

For $a, b \in \mathbb{R}$,

$$\frac{\partial u}{\partial t} + (at+b)\frac{\partial u}{\partial x} = 0 \quad \text{with } u(0,x) = u_0(x).$$

Solution 2.

$$\frac{D}{Dt}u(t,x(t)) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t}.$$

So, we have two ODEs:

$$① \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = at + b \\ x(0) = x_0 \end{cases} \qquad ② \begin{cases} \dfrac{Du}{Dt} = 0 \\ u(0,x(0)) = u_0(x_0). \end{cases}$$

From ①,

$$x(t) = \frac{a}{2}t^2 + bt + x_0$$
$$x_0 = x - \frac{a}{2}t^2 - bt.$$

From ②, $\frac{Du}{Dt} = 0$, so u is a constant. By the IC,

$$\begin{aligned} u(t, x(t)) &= u_0(x_0) \\ u(t, x) &= u_0\Big(x - \frac{a}{2}t^2 - bt\Big). \end{aligned}$$

□

2.2.2 Presence of a Forcing Term

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \gamma u(t, x) = f(t, x), \quad \text{with } u(0, x) = u_0(x).$$

- $\gamma = 0$:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = f(t, x), \quad \text{with } u(0, x) = u_0(x).$$

Solution 3.

Two ODEs:

$$① \begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = c \\ x(0) = x_0 \end{cases} \qquad ② \begin{cases} \frac{Du}{Dt} = f(t, x(t)) \\ u(0, x(0)) = u_0(x_0). \end{cases}$$

From ①,

$$x(t) = ct + x_0 \implies x_0 = x - ct.$$

Solving ②:

$$\begin{aligned} \int_0^t \frac{Du}{Dt}\,\mathrm{d}t &= \int_0^t f(s, x(s))\,\mathrm{d}s \\ u(t, x(t)) - u(0, x(0)) &= \int_0^t f(s, x(s))\,\mathrm{d}s && \text{[Fund. Thm. of Calculus]} \end{aligned}$$

Note that $x(t) = ct + x_0$. So, $x(s) = cs + x_0$. Then,

$$\begin{aligned} u(t, x(t)) - u_0(x_0) &= \int_0^t f(s, cs + x_0)\,\mathrm{d}s && \text{[Use IC]} \\ u(t, x(t)) &= u_0(x_0) + \int_0^t f(s, cs + x_0)\,\mathrm{d}s \\ &= u_0(x - ct) + \int_0^t f(s, cs + x - ct)\,\mathrm{d}s && [x_0 = x - ct] \\ &= u_0(x - ct) + \int_0^t f\big(s, x - c(t - s)\big)\,\mathrm{d}s \end{aligned}$$

□

- $f = 0$:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \gamma u(t,x) = 0, \quad \text{with } u(0,x) = u_0(x).$$

Solution 4.

Two ODEs:

$$① \begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = c \\ x(0) = x_0 \end{cases} \qquad ② \begin{cases} \frac{Du}{Dt} + \gamma u = 0 \\ u(0, x(0)) = u_0(x_0). \end{cases}$$

From ①, $x(t) = ct + x_0 \implies x_0 = x - ct$.

From ②: first-order, linear, homogenous ODE. Use integrating factor:

$$\mu = e^{\int \gamma \,\mathrm{d}t} = e^{\gamma t}.$$

Then,

$$\begin{aligned} \underbrace{e^{\gamma t}\frac{Du}{Dt} + e^{\gamma t}\gamma u}_{} &= 0 \\ \int_0^t \frac{D}{Ds}(e^{\gamma s}u)\,\mathrm{d}s &= 0 \\ e^{\gamma t}u(t,x(t)) - \underbrace{e^{\gamma\cdot 0}}_{=1}u(0,x(0)) &= 0 \\ e^{\gamma t}u(t,x(t)) - u(0,x(0)) &= 0 \\ e^{\gamma t}u(t,x(t)) &= u(0,x(0)) = u_0(x_0) \\ u(t,x(t)) &= e^{-\gamma t}u_0(x_0) \\ u(t,x) &= e^{-\gamma t}u_0(x - ct). \end{aligned}$$

□

- $\gamma \neq 0$ and $f \neq 0$:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \gamma u(t,x) = f(t,x), \quad \text{with } u(0,x) = u_0(x).$$

Solution 5.

Two ODEs:

$$① \begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = c \\ x(0) = x_0 \end{cases} \qquad ② \begin{cases} \frac{Du}{Dt} + \gamma u = f(t, x(t)) \\ u(0, x(0)) = u_0(x_0). \end{cases}$$

From ①, $x(t) = ct + x_0 \implies x_0 = x - ct$.

From ②: first-order, linear, non-homogenous ODE. Use integrating factor

$$\mu = e^{\int \gamma \, \mathrm{d}t} = e^{\gamma t}.$$

Then,

$$\begin{aligned}
\underbrace{e^{\gamma t}\frac{Du}{Dt} + e^{\gamma t}\gamma u}_{} &= e^{\gamma t} f(t, x(t)) \\
\int_0^t \frac{D}{Ds}(e^{\gamma s}u)\,\mathrm{d}u &= \int_0^t e^{\gamma s} f(s, x(s))\,\mathrm{d}s \\
e^{\gamma t}u(t, x(t)) - e^{\gamma \cdot 0}u(0, x(0)) &= \int_0^t e^{\gamma s} f(s, cs + x_0)\,\mathrm{d}s \\
e^{\gamma t}u(t, x(t)) - u_0(x_0) &= \int_0^t e^{\gamma s} f(s, cs + x_0)\,\mathrm{d}x \\
e^{\gamma t}u(t, x(t)) &= u_0(x_0) + \int_0^t e^{\gamma s} f(s, cs + x_0)\,\mathrm{d}s \\
e^{\gamma t}u(t, x(t)) &= u_0(x - ct) + \int_0^t e^{\gamma s} f(s, cs + x - ct)\,\mathrm{d}s \\
e^{\gamma t}u(t, x(t)) &= u_0(x - ct) + \int_0^t e^{\gamma s} f\big(s, x - c(t - s)\big)\,\mathrm{d}s \\
u(t, x(t)) &= e^{-\gamma t}u_0(x - ct) + e^{-\gamma t}\int_0^t e^{\gamma s} f\big(s, x - c(t - s)\big)\,\mathrm{d}s
\end{aligned}$$

□

Summary

For the PDE

$$\begin{cases} \dfrac{\partial u}{\partial t} + c(t, x)\dfrac{\partial u}{\partial x} + \gamma(t, x)u = f(t, x) \\ u(0, x) = u_0(x), \end{cases}$$

we form two ODEs:

$$① \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = c(t, x) \\ x(0) = x_0 \end{cases} \qquad ② \begin{cases} \dfrac{Du}{Dt} + \gamma(t, x(t))u = f(t, x(t)) \\ u(0, x(0)) = u_0(x_0). \end{cases}$$

Equation ② is linear for sure. We use integrating factor to solve it.

Example 2.2.2

$$\frac{\partial u}{\partial t} + 3\frac{\partial u}{\partial x} - 2u = \sin(x - t) \quad \text{with } u(0, x) = x^2.$$

Solution 6.

Two ODEs:

$$① \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = 3 \\ x(0) = x_0 \end{cases} \qquad ② \begin{cases} \dfrac{Du}{Dt} - 2u = \sin(x-t) \\ u(0, x(0)) = u_0(x_0). \end{cases}$$

①: $x = 3t + x_0 \implies x_0 = x - 3t$. ②: Integrating factor.

$$\mu(t) = e^{\int -2\,\mathrm{d}t} = e^{-2t}.$$

Then,

$$\begin{aligned} \frac{D}{Dt}\left[e^{-2t}u\right] &= e^{-2t}\sin(x(t) - t) \\ \int_0^t \frac{D}{Ds}\left[e^{-2s}u\right]\mathrm{d}s &= \int_0^t e^{-2s}\sin(x(s) - s)\,\mathrm{d}s \\ e^{-2t}u(t, x(t)) - u(0, x(0)) &= \int_0^t e^{-2s}\sin(3s + x_0 - s)\,\mathrm{d}s \\ e^{-2t}u(t, x(t)) &= u_0(x_0) + \int_0^t e^{-2s}\sin(2s + x_0)\,\mathrm{d}s \end{aligned}$$

Lets solve this integral:

$$\begin{aligned} \int e^{-2s}\sin(2s+x_0)\,\mathrm{d}s &= -\frac{1}{2}e^{-2s}\sin(2s+x_0) - \int \left(-\frac{1}{2}e^{-2s}\right)2\cos(2s+x_0)\,\mathrm{d}s \\ &= -\frac{1}{2}e^{-2s}\sin(2s+x_0) + \int e^{-2s}\cos(2s+x_0)\,\mathrm{d}s \\ &= -\frac{1}{2}e^{-2s}\sin(2s+x_0) - \frac{1}{2}e^{-2s}\cos(2s+x_0) \\ &\quad - \int \left(-\frac{1}{2}\right)e^{-2s}(-2)\sin(2s+x_0)\,\mathrm{d}s \\ \int e^{-2s}\sin(2s+x_0)\,\mathrm{d}s &= -\frac{1}{2}e^{-2s}\sin(2s+x_0) - \frac{1}{2}e^{-2s}\cos(2s+x_0) \\ &\quad - \int e^{-2s}\sin(2s+x_0)\,\mathrm{d}s \\ 2\int e^{-2s}\sin(2s+x_0)\,\mathrm{d}s &= -\frac{1}{2}e^{-2s}[\sin(2s+x_0) + \cos(2s+x_0)] \\ \int e^{-2s}\sin(2s+x_0)\,\mathrm{d}s &= -\frac{1}{4}e^{-2s}[\sin(2s+x_0) + \cos(2s+x_0)] \end{aligned}$$

So,

$$\begin{aligned} e^{-2t}u(t,x(t)) &= u_0(x_0) - \frac{1}{4}e^{-2t}[\sin(2t+x_0)+\cos(2t+x_0)] + \frac{1}{4}e^{0}[\sin(x_0)+\cos(x_0)] \\ &= (x-3t)^2 - \frac{1}{4}e^{-2t}[\sin(x-t)+\cos(x-t)] + \frac{1}{4}[\sin(x-3t)+\cos(x-3t)] \\ u(t,x) &= e^{2t}(x-3t)^2 - \frac{1}{4}[\sin(x-t)+\cos(x-t)] + \frac{1}{4}e^{2t}[\sin(x-3t)+\cos(x-3t)]. \end{aligned}$$

□

Example 2.2.3

$$\frac{\partial u}{\partial t} + 2(t^2+1)\frac{\partial u}{\partial x} = 0 \quad \text{with } u(0,x) = e^x.$$

Solution 7.

Two ODEs:

$$① \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = 2(t^2+1) \\ x(0) = x_0 \end{cases} \qquad ② \begin{cases} \dfrac{Du}{Dt} = 0 \\ u(0,x(0)) = e_0^x. \end{cases}$$

①:

$$\begin{aligned} x(t) &= \int 2(t^2+1)\,\mathrm{d}t + x_0 \\ &= 2\left[\frac{1}{3}t^3 + t\right] + x_0 \\ &= \frac{2}{3}t^3 + 2t + x_0 \\ x_0 &= x - \frac{2}{3}t^3 - 2t \end{aligned}$$

②: $u(t,x(t))$ is a constant. So,

$$u(t,x(t)) = e^{x_0} \implies u(t,x) = e^{x-2/3\cdot t^3-2t}.$$

□

Example 2.2.4

$$\frac{\partial u}{\partial t} + \frac{1}{3x^2}\frac{\partial u}{\partial x} + 2u = t \quad \text{with } u(0,x) = u_0(x).$$

Solution 8.

Two ODEs:

$$\text{①}\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=\dfrac{1}{3x^2}\\ x(0)=x_0\end{cases}\qquad \text{②}\begin{cases}\dfrac{Du}{Dt}+2u=t\\ u(0,x(0))=u_0(x_0).\end{cases}$$

①:

$$\begin{aligned}\int 3x^2\,\mathrm{d}x&=\int \mathrm{d}t+C && \textit{[first-order, separable]}\\ 3\cdot\frac{1}{3}x^3&=t+C\\ x^3&=t+C.\end{aligned}$$

When $t=0$, $x(0)=x_0$. So, $x_0^3=0+C \implies C=x_0^3$. So,

$$\begin{aligned}x^3&=t+x_0^3\\ x_0^3&=x^3-t\\ x_0&=(x^3-t)^{1/3}.\end{aligned}$$

②: $\dfrac{D}{Dt}u+2u=t$. Integrating factor:

$$\mu(t)=e^{\int 2\,\mathrm{d}t}=e^{2t}.$$

Then,

$$\begin{aligned}e^{2t}\frac{Du}{Dt}+2e^{2t}u&=e^{2t}t\\ \frac{D}{Dt}\left[e^{2t}u\right]&=e^{2t}\cdot t\\ \int_0^t e^{2s}u\,\mathrm{d}s&=\int_0^t e^{2s}\cdot s\,\mathrm{d}s\\ e^{2t}u(t,x(t))-e^{2\cdot 0}u(0,x(0))&=\left[\frac{1}{2}e^{2s}s\right]_0^t-\int_0^t\frac{1}{2}e^{2s}\,\mathrm{d}s=\frac{1}{2}e^{2t}\cdot t-\left[\frac{1}{4}e^{2s}\right]_0^t\\ e^{2t}u(t,x(t))-u_0(x_0)&=\frac{1}{2}e^{2t}\cdot t-\frac{1}{4}e^{2t}+\frac{1}{4}\\ e^{2t}u(t,x(t))&=u_0(x_0)+\frac{1}{2}e^{2t}\cdot t-\frac{1}{4}e^{2t}+\frac{1}{4}\\ e^{2t}u(t,x)&=u_0\big((x^3-t)^{1/3}\big)+\frac{1}{2}e^{2t}\cdot t-\frac{1}{4}e^{2t}+\frac{1}{4}\\ u(t,x)&=e^{-2t}u_0\big((x^3-t)^{1/3}\big)+\frac{1}{2}t-\frac{1}{4}+\frac{1}{4}e^{-2t}.\end{aligned}$$

□

Example 2.2.5

$$t\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 0 \quad \text{with } u(1,x) = \sin x.$$

Solution 9.

Two ODEs:

$$① \begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x}{t} \\ x(1) = x_1 \end{cases} \qquad ② \begin{cases} \frac{Du}{Dt} = 0 \\ u(1, x(1)) = \sin x_1. \end{cases}$$

①:

$$\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{1}{t}x = 0 \qquad \textit{[first-order, linear, homo]}$$
$$x = Ce^{-\int -1/t\,\mathrm{d}t} = Ce^{\ln(t)} = Ct.$$

At $t = 1$, $x(1) = C = x_1$. So, $x = x_1 t \implies x_1 = \frac{x}{t}$.

②: $u(t, x(t))$ is a constant.

$$u(t, x(t)) = \sin x_1$$
$$u(t,x) = \sin\left(\frac{x}{t}\right).$$

□

2.3 System of First Order PDEs

$$\begin{cases} \frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} + Mu = f, \quad A, M \in \mathbb{R}^{n\times n} \\ u_0(0,x) = u_0(x), \end{cases}$$

where

$$u(t,x) = \begin{bmatrix} u_1(t,x) \\ u_2(t,x) \\ \vdots \\ u_n(t,x) \end{bmatrix} \in \mathbb{R}^n, \ \frac{\partial u}{\partial t} = \begin{bmatrix} \partial u_1/\partial t \\ \partial u_2/\partial t \\ \vdots \\ \partial u_n/\partial t \end{bmatrix} \in \mathbb{R}^n, \ \frac{\partial u}{\partial x} = \begin{bmatrix} \partial u_1/\partial x \\ \partial u_2/\partial x \\ \vdots \\ \partial u_n/\partial x \end{bmatrix} \in \mathbb{R}^n, \ f(t,x) = \begin{bmatrix} f_1(t,x) \\ f_2(t,x) \\ \vdots \\ f_n(t,x) \end{bmatrix}.$$

Suppose $n = 2$. Then,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Set $M = 0, f = 0$. Then, we have a system of two PDEs:

$$\begin{cases} \text{[PDEs]} & \begin{cases} \dfrac{\partial u_1}{\partial t} + a_{11}\dfrac{\partial u_1}{\partial x} + a_{12}\dfrac{\partial u_2}{\partial x} = 0 \\ \\ \dfrac{\partial u_2}{\partial t} + a_{21}\dfrac{\partial u_1}{\partial x} + a_{12}\dfrac{\partial u_2}{\partial x} = 0 \end{cases} \\ \\ \text{[ICs]} & \begin{cases} u_1(0,x) = u_{1,0}(x) \\ u_2(0,x) = u_{2,0}(x). \end{cases} \end{cases}$$

But... we can't solve them yet. If A is diagonal, then $a_{12} = a_{21} = 0$, we are back to the transport equation, and we just need to solve two transport equations. So, our job is to diagonalize A. By diagonalization, we have

$$A = VDV^{-1},$$

where

$$V = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}.$$

Then,

$$\begin{aligned} \frac{\partial u}{\partial t} + VDV^{-1}\frac{\partial u}{\partial x} &= 0 \\ V^{-1}\frac{\partial u}{\partial t} + \underbrace{V^{-1}V}_{=I} DV^{-1}\frac{\partial u}{\partial x} &= 0 \\ V^{-1}\frac{\partial u}{\partial t} + DV^{-1}\frac{\partial u}{\partial x} &= 0 \qquad \text{[\textit{Change of Variable:} } w = V^{-1}u\text{]} \\ \frac{\partial w}{\partial t} + D\frac{\partial w}{\partial x} &= 0. \end{aligned}$$

This is easy to solve! After finding w, we have $w = V^{-1}u \implies u = Vw$.

Definition 2.3.1 (Riemann Variable). The auxiliary variables w are called *Riemann variables* or *invariants.*

2.3.1 Review: How to find the diagonalization – Eigenvalue Problem

Find a pair (λ, v) $s.t.$ $Av = \lambda v$ with $v \neq 0$. Then,

$$Av = \lambda v \implies \underbrace{(\lambda I - A)}_{I} v = 0 \implies Bv = 0$$

Since $v \neq 0$ by assumption, it must be that B is singular. That is,

$$\det(B) = \det(\lambda I - A) = 0.$$

This is equivalent to finding roots to the characteristic polynomial

$$\beta_0 \lambda^m + \beta_1 \lambda^{m-1} + \cdots + \beta_m = 0.$$

By Fundamental Theorem of Algebra, characteristic polynomial has exactly m roots (and they can occur multiple times) $\in \mathbb{C}$. The number of times they occur is called *multiplicity*.

For $n = 2$,

$$\det(\lambda I - A) = a\lambda^2 + b\lambda + c = 0.$$

Solution formula is

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Given $\lambda_{1,2}$, we can also find corresponding eigenvectors v_1, v_2. Then,

$$V = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}.$$

For 2×2 matrix,

$$V = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies V^{-1} = \frac{1}{\det(V)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example 2.3.2

$$A = \begin{bmatrix} 3 & -3 \\ -4 & -1 \end{bmatrix}$$

Solution 1.

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & -3 \\ -4 & -1 \end{bmatrix} = \begin{bmatrix} \lambda - 3 & 3 \\ 4 & \lambda + 1 \end{bmatrix}.$$

$$\begin{aligned} \det(\lambda I - A) = (\lambda - 3)(\lambda + 1) - 12 &= 0 \\ \lambda^2 - 2\lambda - 3 - 12 &= 0 \\ \lambda^2 - 2\lambda - 15 &= 0 \\ (\lambda - 5)(\lambda + 3) &= 0 \implies \lambda_1 = 5, \lambda_2 = -3. \end{aligned}$$

- Consider $Ax = 5x$.

$$\begin{cases} 3x_1 - 3x_2 &= 5x_1 \\ -4x_1 - x_2 &= 5x_2 \end{cases} \implies \begin{cases} 2x_1 = -3x_2 \\ -4x_1 = 6x_2 \end{cases} \implies x = \begin{bmatrix} -3/2x_2 \\ x_2 \end{bmatrix}.$$

So, the eigenvector corresponding to $\lambda_1 = 5$ is $\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$.

- Consider $Ax = -3x$.

$$\begin{cases} 3x_1 - 3x_2 = -3x_1 \\ -4x_1 - x_2 = -3x_2 \end{cases} \implies \begin{cases} 6x_1 = 3x_2 \\ -4x_1 = -2x_2 \end{cases} \implies x = \begin{bmatrix} 1/2x_2 \\ x_2 \end{bmatrix}.$$

So, the eigenvector corresponding to $\lambda_2 = -3$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Hence, the diagonalization of A is $A = VDV^{-1}$, where

$$V = \begin{bmatrix} -3/2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}.$$

Then,

$$\begin{aligned} V^{-1} = \frac{1}{\det(V)} \begin{bmatrix} 2 & -1 \\ -1 & -3/2 \end{bmatrix} &= \frac{1}{-3-1} \begin{bmatrix} 2 & -1 \\ -1 & -3/2 \end{bmatrix} \\ &= \frac{1}{-4} \begin{bmatrix} 2 & -1 \\ -1 & -3/2 \end{bmatrix} \\ &= \begin{bmatrix} -1/2 & 1/4 \\ 1/4 & 3/8 \end{bmatrix}. \end{aligned}$$

□

2.3.2 Worked Examples

Example 2.3.3

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0 \quad \text{with } u(0,x) = \begin{bmatrix} x^3 \\ x^2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 3 & -3 \\ -4 & -1 \end{bmatrix}.$$

Solution 2.

We have found: $\lambda_1 = -3$, $\lambda_2 = 5$, $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $v_2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Then,

$$D = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}.$$

Since $\det(V) = 2 + 6 = 8$, we have

$$V^{-1} = \frac{1}{\det(V)} \begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}.$$

So,

$$w_0 = V^{-1} u_0 = \frac{1}{8} \begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x^3 \\ x^2 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 2x^3 + 3x^2 \\ -2x^3 + x^2 \end{bmatrix} = \begin{bmatrix} w_{01}(x) \\ w_{02}(x) \end{bmatrix}.$$

Then, we have two PDEs:

- $$\begin{cases} \dfrac{\partial w_1}{\partial t} - 3\dfrac{\partial w_1}{\partial x} = 0 \\ \\ w_{01}(x) = \dfrac{1}{8}(2x^3 + 3x^2) \end{cases}$$

  We have two ODEs:

  $$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -3 \\ x(0) = x_{01} \end{cases} \qquad \begin{cases} \dfrac{Dw_1}{Dt} = 0 \\ w_{01}(0, x(0)) = \dfrac{1}{8}(2x_{01}^3 + 3x_{01}^2). \end{cases}$$

  Then, we have $x(t) = -3t + x_{01} \implies x_{01} = x + 3t$. Meanwhile,

  $$\begin{aligned} w_1(t, x(t)) &= w_{01}(x_{01}) \\ &= \frac{1}{8}\big(2x_{01}^3 + 3x_{01}^2\big) \\ w_1(t, x) &= \frac{1}{8}\big(2(x + 3t)^3 + 3(x + 3t)^2\big). \end{aligned}$$

- $$\begin{cases} \dfrac{\partial w_2}{\partial t} + 5\dfrac{\partial w_2}{\partial x} = 0 \\ \\ w_{02}(x) = \dfrac{1}{8}(-2x^3 + x^2) \end{cases}$$

We have two ODEs:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = 5 \\ x(0) = x_{02} \end{cases} \qquad \begin{cases} \frac{Dw_2}{Dt} = 0 \\ w_{02}(0, x(0)) = \frac{1}{8}(-2x_{02}^3 + x_{02}^2). \end{cases}$$

Then, we have $x(t) = 5t + x_{02} \implies x_{02} = x - 5t$. Meanwhile,

$$\begin{aligned} w_2(t, x(t)) &= w_{02}(x_{02}) \\ &= \frac{1}{8}\left(-2x_{02}^3 + x_{02}^2\right) \\ w_2(t, x) &= \frac{1}{8}\left(-2(x-5t)^3 + (x-5t)^2\right). \end{aligned}$$

Finally, the solution is

$$\begin{aligned} u = Vw &= \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\ &= \begin{bmatrix} w_1 - 3w_2 \\ 2w_1 + 2w_2 \end{bmatrix} \\ &= \begin{bmatrix} 1/8(2(x+3t)^3 + 3(x+3t)^2) - 3/8(-2(x-5t)^3 + (x-5t)^2) \\ 1/4(2(x+3t)^3 + 3(x+3t)^2) + 1/4(-2(x-5t)^3 + (x-5t)^2) \end{bmatrix}. \end{aligned}$$

□

Example 2.3.4 Non-homogenous System, $f \neq 0$

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = f \quad \text{with } u(0, x) = \begin{bmatrix} x^3 \\ x^2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 3 & -3 \\ -4 & -1 \end{bmatrix} \quad \text{and} \quad f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Solution 3.
Suppose $A = VDV^{-1}$. Then,

$$\begin{aligned} \frac{\partial u}{\partial t} + VDV^{-1}\frac{\partial u}{\partial x} &= f \\ V^{-1}\frac{\partial u}{\partial t} + \underbrace{V^{-1}V}_{=I} DV^{-1}\frac{\partial u}{\partial x} &= V^{-1}f && \textit{[Multiply by } V^{-1}\textit{]} \\ V^{-1}\frac{\partial u}{\partial t} + DV^{-1}\frac{\partial u}{\partial x} = V^{-1}f \implies \frac{\partial w}{\partial t} + D\frac{\partial w}{\partial x} &= V^{-1}f && [w = V^{-1}u] \end{aligned}$$

Recall from previous example:

$$V = \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}, \quad V^{-1} = \frac{1}{8}\begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}, \quad \text{and } D = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}.$$

Then,

$$V^{-1}f = \frac{1}{8}\begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/8 \\ 1/8 \end{bmatrix}.$$

Then, we have two PDEs:

- $$\begin{cases} \dfrac{\partial w_1}{\partial t} - 3\dfrac{\partial w_1}{\partial x} = \dfrac{3}{8} \\ w_{01}(x) = \dfrac{1}{8}(2x^3 + 3x^2) \end{cases}$$

  We have two ODEs:

  $$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -3 \\ x(0) = x_{01} \end{cases} \qquad \begin{cases} \dfrac{Dw_1}{Dt} = \dfrac{3}{8} \\ w_{01}(0, x(0)) = \dfrac{1}{8}(2x_{01}^3 + 3x_{01}^2). \end{cases}$$

  Then, we have $x(t) = -3t + x_{01} \implies x_{01} = x + 3t$. Meanwhile,

  $$\begin{aligned} \frac{Dw_1}{Dt} = \frac{3}{8} \implies w_1(t, x(t)) &= \frac{3}{8}t + w_{01}(x_{01}) \\ &= \frac{3}{8}t + \frac{1}{8}\big(2x_{01}^3 + 3x_{01}^2\big) \\ w_1(t,x) &= \frac{3}{8}t + \frac{1}{8}\big(2(x+3t)^3 + 3(x+3t)^2\big). \end{aligned}$$

- $$\begin{cases} \dfrac{\partial w_2}{\partial t} + 5\dfrac{\partial w_2}{\partial x} = \dfrac{1}{8} \\ w_{02}(x) = \dfrac{1}{8}(-2x^3 + x^2) \end{cases}$$

  We have two ODEs:

  $$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = 5 \\ x(0) = x_{02} \end{cases} \qquad \begin{cases} \dfrac{Dw_2}{Dt} = \dfrac{1}{8} \\ w_{02}(0, x(0)) = \dfrac{1}{8}(-2x_{02}^3 + x_{02}^2). \end{cases}$$

Then, we have $x(t) = 5t + x_{02} \implies x_{02} = x - 5t$. Meanwhile,

$$\begin{aligned}\frac{Dw_2}{Dt} = \frac{1}{8} \implies w_2(t, x(t)) &= \frac{1}{8}t + w_{02}(x_{02}) \\ &= \frac{1}{8}t + \frac{1}{8}\left(-2x_{02}^3 + x_{02}^2\right) \\ w_2(t,x) &= \frac{1}{8}t + \frac{1}{8}\left(-2(x-5t)^3 + (x-5t)^2\right).\end{aligned}$$

Finally, the solution is

$$\begin{aligned}u = Vw &= \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\ &= \begin{bmatrix} w_1 - 3w_2 \\ 2w_1 + 2w_2 \end{bmatrix} && [\textit{Plug-in } w_1 \textit{ and } w_2]\end{aligned}$$

□

Now, consider the full system:

$$\begin{cases} \dfrac{\partial u}{\partial t} + A\dfrac{\partial u}{\partial x} + Mu = f \\ u(0,x) = u_0(x) \end{cases}$$

Recall: if $A = VDV^{-1}$, then

$$\begin{aligned}\frac{\partial u}{\partial t} + VDV^{-1}\frac{\partial u}{\partial x} + Mu &= f \\ V^{-1}\frac{\partial u}{\partial t} + \underbrace{V^{-1}V}_{=I}DV^{-1}\frac{\partial u}{\partial x} + V^{-1}Mu &= V^{-1}f \\ V^{-1}\frac{\partial u}{\partial t} + DV^{-1}\frac{\partial u}{\partial x} + V^{-1}Mu &= V^{-1}f.\end{aligned}$$

- If M is the identity matrix, $M = I$. Then,

$$\begin{aligned}V^{-1}\frac{\partial u}{\partial t} + DV^{-1}\frac{\partial u}{\partial x} + V^{-1}Iu &= V^{-1}f \\ V^{-1}\frac{\partial u}{\partial t} + DV^{-1}\frac{\partial u}{\partial x} + V^{-1}u &= V^{-1}f \\ \frac{\partial w}{\partial t} + D\frac{\partial w}{\partial x} + w &= V^{-1}f && [w = V^{-1}u]\end{aligned}$$

- If M is a multiple of the identity matrix, $M = \alpha I$. Then,

$$\begin{aligned} V^{-1}\frac{\partial u}{\partial t} + DV^{-1}\frac{\partial u}{\partial x} + V^{-1}(\alpha I)u &= V^{-1}f \\ V^{-1}\frac{\partial u}{\partial t} + DV^{-1}\frac{\partial u}{\partial x} + \alpha V^{-1}u &= V^{-1}f \\ \frac{\partial w}{\partial t} + D\frac{\partial w}{\partial x} + \alpha w &= V^{-1}f && [w = V^{-1}u] \end{aligned}$$

- If M is diagonalized by V, (that is, $M = V\widetilde{M}V^{-1}$, where $\widetilde{M}$ is diagonal), then

$$\begin{aligned} V^{-1}\frac{\partial u}{\partial t} + DV^{-1}\frac{\partial u}{\partial x} + \underbrace{V^{-1}(V}_{=I}\widetilde{M}V^{-1})u &= V^{-1}f \\ V^{-1}\frac{\partial u}{\partial t} + DV^{-1}\frac{\partial u}{\partial x} + \widetilde{M}V^{-1}u &= V^{-1}f \\ \frac{\partial w}{\partial t} + D\frac{\partial w}{\partial x} + \widetilde{M}w &= V^{-1}f && [w = V^{-1}u] \end{aligned}$$

- Other cases: Numerical Methods.

Example 2.3.5

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = f \quad \text{with } u(0,x) = \begin{bmatrix} x^3 \\ x^2 \end{bmatrix}, \; A = \begin{bmatrix} 3 & -3 \\ -4 & -1 \end{bmatrix}, \; M = \begin{bmatrix} 5/4 & 3/8 \\ 1/2 & 7/4 \end{bmatrix}, \text{ and } f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Solution 4.
Firstly, let's verify $V^{-1}MV$ is diagonal. Recall:

$$V = \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}, \quad V^{-1} = \frac{1}{8}\begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}, \quad \text{and } D = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}.$$

Then,

$$\begin{aligned} \widetilde{M} = V^{-1}MV &= \frac{1}{8}\begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5/4 & 3/8 \\ 1/2 & 7/4 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix} \\ &= \frac{1}{8}\begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 4 & 2 \end{bmatrix} \\ &= \frac{1}{8}\begin{bmatrix} 16 & 0 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Then, we aim to solve

$$\frac{\partial w}{\partial t} + D\frac{\partial w}{\partial x} + \widetilde{M}w = V^{-1}f,$$

where $w = V^{-1}u$. Then, we have two PDEs to solve:

- $$\begin{cases} \dfrac{\partial w_1}{\partial t} - 3\dfrac{\partial w_1}{\partial x} + 2w_1 = \dfrac{3}{8} \\ \\ w_{01}(0,x) = \dfrac{1}{8}(2x^3 + 3x^2) \end{cases}$$

  1. $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -3 \\ x(0) = x_{01} \end{cases} \implies$ We get $x(t) = -3t + x_{01} \implies x_{01} = x + 3t$.
  2. $\begin{cases} \dfrac{Dw_1}{Dt} + 2w_1 = \dfrac{3}{8} \\ \\ w_{01}(0, x(0)) = \dfrac{1}{8}(2x_{01}^3 + 3x_{01}^2). \end{cases}$

     Apply integrating factor:

     $$\mu = e^{\int 2\,\mathrm{d}t} = e^{2t}.$$

     Then,

     $$\begin{aligned} \left[e^{2s}w_1(s,x(s))\right]_0^t &= \left[e^{2s}\frac{3}{8}\right]_0^t \\ e^{2t}w_1(t,x(t)) - w_1(0,x(0)) &= \frac{3}{8}\left(e^{2t}-1\right) \\ e^{2t}w_1(t,x(t)) &= \frac{3}{8}\left(e^{2t}-1\right) + w_1(0,x(0)) \\ w_1(t,x(t)) &= \frac{3}{8}\left(1-e^{-2t}\right) + e^{-2t}w_1(0,x(0)) \\ &= \frac{3}{8}\left(1-e^{-2t}\right) + e^{-2t}\frac{1}{8}\left(2x_{01}^3 + 3x_{01}^2\right) \\ \boxed{w_1(t,x) = \frac{3}{8}\left(1-e^{-2t}\right) + \frac{1}{8}e^{-2t}\left(2(x+3t)^3 + 3(x+3t)^2\right)} \end{aligned}$$

- $$\begin{cases} \dfrac{\partial w_2}{\partial t} + 5\dfrac{\partial w_2}{\partial x} + w_2 = \dfrac{1}{8} \\ \\ w_{02}(0,x) = \dfrac{1}{8}(-2x^3 + x^2) \end{cases}$$

  1. $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = 5 \\ x(0) = x_{02} \end{cases} \implies$ We get $x(t) = 5t + x_{02} \implies x_{02} = x - 5t$.

2. $$\begin{cases} \dfrac{Dw_2}{Dt} + w_2 = \dfrac{1}{8} \\ \\ w_{02}(0, x(0)) = \dfrac{1}{8}(-2x_{02}^3 + x_{02}^2). \end{cases}$$

Apply integrating factor:

$$\mu = e^{\int 1\,\mathrm{d}t} = e^t.$$

Then,

$$\begin{aligned} \Big[e^s w_2(s, x(s))\Big]_0^t &= \Big[e^s \frac{1}{8}\Big]_0^t \\ e^t w_2(t, x(t)) - w_2(0, x(0)) &= \frac{1}{8}\big(e^t - 1\big) \\ e^t w_2(t, x(t)) &= \frac{1}{8}\big(e^t - 1\big) + w_2(0, x(0)) \\ w_2(t, x(t)) &= \frac{1}{8}\big(1 - e^{-t}\big) + e^{-t} w_2(0, x(0)) \\ &= \frac{1}{8}\big(1 - e^{-t}\big) + e^{-t}\frac{1}{8}\big(-2x_{02}^3 + x_{02}^2\big) \\ \boxed{w_2(t, x) = \frac{1}{8}\big(1 - e^{-t}\big) + \frac{1}{8}e^{-t}\big(-2(x - 5t)^3 + (x - 5t)^2\big)} \end{aligned}$$

Finally, the solution is

$$\begin{aligned} u = Vw &= \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\ &= \begin{bmatrix} w_1 - 3w_2 \\ 2w_1 + 2w_2 \end{bmatrix} \qquad \textit{[Plug-in } w_1 \textit{ and } w_2\textit{]} \end{aligned}$$

□

Example 2.3.6 Connection with Wave Equation I

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0, \quad \text{with} \quad u(0, x) = \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix} \text{ and } A = \begin{bmatrix} 0 & -1 \\ -\gamma^2 & 0 \end{bmatrix}.$$

Solution 5.

$$\det(A) = \det \begin{bmatrix} -\lambda & -1 \\ -\gamma^2 & -\lambda \end{bmatrix} = \lambda^2 - \gamma^2 = 0 \implies \lambda_1 = \gamma,\ \lambda_2 = -\gamma.$$

- $\lambda_1 = -\gamma$:
$$\begin{bmatrix} \gamma & -1 \\ -\gamma^2 & \gamma \end{bmatrix} \begin{bmatrix} v^{(1)} \\ v^{(2)} \end{bmatrix} = 0 \implies \gamma v^{(1)} - v^{(2)} = 0 \implies v_1 = \begin{bmatrix} 1 \\ \gamma \end{bmatrix}$$

- $\lambda_2 = \gamma$:
$$\begin{bmatrix} -\gamma & -1 \\ -\gamma^2 & -\gamma \end{bmatrix} \begin{bmatrix} v^{(1)} \\ v^{(2)} \end{bmatrix} = 0 \implies -\gamma v^{(1)} - v^{(2)} = 0 \implies v_2 = \begin{bmatrix} 1 \\ -\gamma \end{bmatrix}.$$

So,
$$D = \begin{bmatrix} -\gamma & 0 \\ 0 & \gamma \end{bmatrix}, \quad \text{and} \quad V = \begin{bmatrix} 1 & 1 \\ \gamma & -\gamma \end{bmatrix}.$$

Since
$$\det(V) = \det \begin{bmatrix} 1 & 1 \\ \gamma & -\gamma \end{bmatrix} = -2\gamma,$$

we know
$$V^{-1} = \frac{1}{\det(V)} \begin{bmatrix} -\gamma & -1 \\ -\gamma & 1 \end{bmatrix} = -\frac{1}{2\gamma} \begin{bmatrix} -\gamma & -1 \\ -\gamma & 1 \end{bmatrix} = \frac{1}{2\gamma} \begin{bmatrix} \gamma & 1 \\ \gamma & -1 \end{bmatrix}.$$

Then, The initial condition is

$$w_0 = \begin{bmatrix} w_{01} \\ w_{02} \end{bmatrix} = \frac{1}{2\gamma} \begin{bmatrix} \gamma & 1 \\ \gamma & -1 \end{bmatrix} \begin{bmatrix} u_{01} \\ u_{02} \end{bmatrix} = \frac{1}{2\gamma} \begin{bmatrix} \gamma & 1 \\ \gamma & -1 \end{bmatrix} \begin{bmatrix} \sin x \\ \cos x \end{bmatrix} = \frac{1}{2\gamma} \begin{bmatrix} \gamma \sin x + \cos x \\ \gamma \sin x - \cos x \end{bmatrix}.$$

So, we have two PDEs to solve:

- $$\begin{cases} \dfrac{\partial w_1}{\partial t} - \gamma \dfrac{\partial w_1}{\partial x} = 0 \\ w_1(0, x) = \dfrac{1}{2\gamma}(\gamma \sin x + \cos x) \end{cases}$$

1. $$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -\gamma \\ x(0) = x_{01} \end{cases} \implies x(t) = -\gamma t + x_{01} \implies x_{01} = x + \gamma t.$$

2. $\begin{cases} \dfrac{Dw_1}{Dt} = 0 \\ w_1(0, x(0)) = \dfrac{1}{2\gamma}(\gamma \sin x_{01} + \cos x_{01}) \end{cases}$

$$\begin{aligned} w_1(t, x(t)) &= w_1(0, x_{01}) \\ &= \frac{1}{2\gamma}(\gamma \sin x_{01} + \cos x_{01}) \\ w_1(t, x) &= \frac{1}{2\gamma}(\gamma \sin(x + \gamma t) + \cos(x + \gamma t)) \end{aligned}$$

- $\begin{cases} \dfrac{\partial w_2}{\partial t} + \gamma \dfrac{\partial w_2}{\partial x} = 0 \\ w_2(0, x) = \dfrac{1}{2\gamma}(\gamma \sin x - \cos x) \end{cases}$

1. $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \gamma \\ x(0) = x_{02} \end{cases} \implies x(t) = \gamma t + x_{02} \implies x_{02} = x - \gamma t.$

2. $\begin{cases} \dfrac{Dw_2}{Dt} = 0 \\ w_2(0, x(0)) = \dfrac{1}{2\gamma}(\gamma \sin x_{02} - \cos x_{02}) \end{cases}$

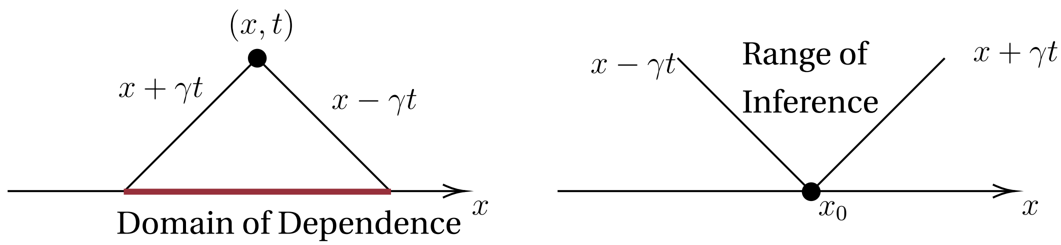
$$\begin{aligned} w_2(t, x(t)) &= w_2(0, x_{02}) \\ &= \frac{1}{2\gamma}(\gamma \sin x_{02} - \cos x_{02}) \\ w_2(t, x) &= \frac{1}{2\gamma}(\gamma \sin(x - \gamma t) - \cos(x - \gamma t)) \end{aligned}$$

So,

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \frac{1}{2\gamma} \begin{bmatrix} \gamma \sin(x + \gamma t) + \cos(x + \gamma t) \\ \gamma \sin(x - \gamma t) - \cos(x - \gamma t) \end{bmatrix}.$$

Then, the final solution is

$$\begin{aligned} u = Vw &= \begin{bmatrix} 1 & 1 \\ \gamma & -\gamma \end{bmatrix} \frac{1}{2\gamma} \begin{bmatrix} \gamma \sin(x + \gamma t) + \cos(x + \gamma t) \\ \gamma \sin(x - \gamma t) - \cos(x - \gamma t) \end{bmatrix} \\ &= \frac{1}{2\gamma} \begin{bmatrix} \gamma \sin(x + \gamma t) + \cos(x + \gamma t) + \gamma \sin(x - \gamma t) - \cos(x - \gamma t) \\ \gamma^2 \sin(x + \gamma t) + \gamma \cos(x + \gamma t) - \gamma^2 \sin(x - \gamma t) + \gamma \cos(x - \gamma t) \end{bmatrix}. \end{aligned}$$



□

Example 2.3.7 Connection with Wave Equation II

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0, \quad \text{with} \quad u(0,x) = \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix}, \ A = \begin{bmatrix} 0 & -1 \\ -\gamma^2 & 0 \end{bmatrix}, \ \text{and } f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Solution 6.

$$D = \begin{bmatrix} -\gamma & 0 \\ 0 & \gamma \end{bmatrix}, \quad \text{and} \quad V = \begin{bmatrix} 1 & 1 \\ \gamma & -\gamma \end{bmatrix}.$$

$$V^{-1} = \frac{1}{2\gamma}\begin{bmatrix} \gamma & 1 \\ \gamma & -1 \end{bmatrix}, \quad w_0 = \frac{1}{2\gamma}\begin{bmatrix} \gamma \sin x + \cos x \\ \gamma \sin x - \cos x \end{bmatrix}.$$

$$\widetilde{f} = V^{-1} f = \frac{1}{2\gamma}\begin{bmatrix} \gamma & 1 \\ \gamma & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2\gamma}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So, we have two PDEs to solve:

- $$\begin{cases} \dfrac{\partial w_1}{\partial t} - \gamma \dfrac{\partial w_1}{\partial x} = \dfrac{1}{2\gamma} \\ w_1(0,x) = \dfrac{1}{2\gamma}(\gamma \sin x + \cos x) \end{cases}$$

1. $$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -\gamma \\ x(0) = x_{01} \end{cases} \implies x(t) = -\gamma t + x_{01} \implies x_{01} = x + \gamma t.$$

2. $$\begin{cases} \dfrac{Dw_1}{Dt} = \dfrac{1}{2\gamma} \\ w_1(0, x(0)) = \dfrac{1}{2\gamma}(\gamma \sin x_{01} + \cos x_{01}) \end{cases}$$

$$\begin{aligned} w_1(t,x) &= \frac{1}{2\gamma} t + \frac{1}{2\gamma}(\gamma \sin(x+\gamma t) + \cos(x+\gamma t)) \\ &= \frac{1}{2\gamma}(\gamma \sin(x+\gamma t) + \cos(x+\gamma t) + t). \end{aligned}$$

- $\begin{cases} \dfrac{\partial w_2}{\partial t} + \gamma \dfrac{\partial w_2}{\partial x} = -\dfrac{1}{2\gamma} \\ w_2(0,x) = \dfrac{1}{2\gamma}(\gamma \sin x - \cos x) \end{cases}$

  1. $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \gamma \\ x(0) = x_{02} \end{cases} \implies x(t) = \gamma t + x_{02} \implies x_{02} = x - \gamma t.$
  2. $\begin{cases} \dfrac{Dw_2}{Dt} = -\dfrac{1}{2\gamma} \\ w_2(0,x(0)) = \dfrac{1}{2\gamma}(\gamma \sin x_{02} - \cos x_{02}) \end{cases}$

$$\begin{aligned} w_2(t,x) &= -\frac{1}{2\gamma} t + \frac{1}{2\gamma}(\gamma \sin(x-\gamma t) - \cos(x-\gamma t)) \\ &= \frac{1}{2\gamma}(\gamma \sin(x-\gamma t) - \cos(x-\gamma t) - t). \end{aligned}$$

So,

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \frac{1}{2\gamma} \begin{bmatrix} \gamma \sin(x+\gamma t) + \cos(x+\gamma t) + t \\ \gamma \sin(x-\gamma t) - \cos(x-\gamma t) - t \end{bmatrix}.$$

Then, the final solution is

$$\begin{aligned} u = Vw &= \begin{bmatrix} 1 & 1 \\ \gamma & -\gamma \end{bmatrix} \frac{1}{2\gamma} \begin{bmatrix} \gamma \sin(x+\gamma t) + \cos(x+\gamma t) + t \\ \gamma \sin(x-\gamma t) - \cos(x-\gamma t) - t \end{bmatrix} \\ &= \frac{1}{2\gamma} \begin{bmatrix} \gamma \sin(x+\gamma t) + \cos(x+\gamma t) + \gamma \sin(x-\gamma t) - \cos(x-\gamma t) \\ \gamma^2 \sin(x+\gamma t) + \gamma \cos(x+\gamma t) - \gamma^2 \sin(x-\gamma t) + \gamma \cos(x-\gamma t) + 2\gamma t \end{bmatrix}. \end{aligned}$$

□

3 Second Order PDE: Unbounded Wave Equation

3.1 Vibrate String

Problem Set-Up Consider a flexible String that is stretched tight between 2 points. The stretching creates a tension T that pulls in both directions at each point along its length. Any other force is negligible. *Write a PDE for this problem.*

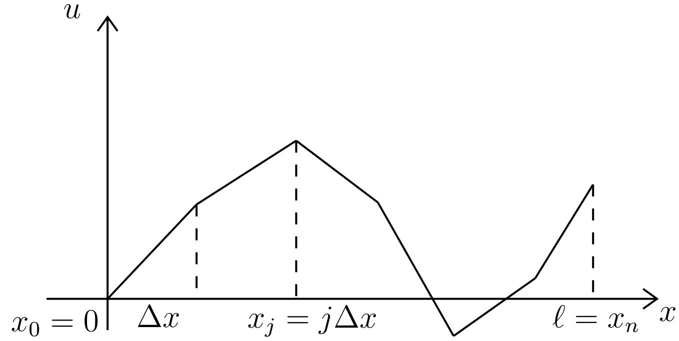
Solution 1.

- Step 1: Parametrize the String: $x \in [0, \ell]$, where ℓ is the length of the string.

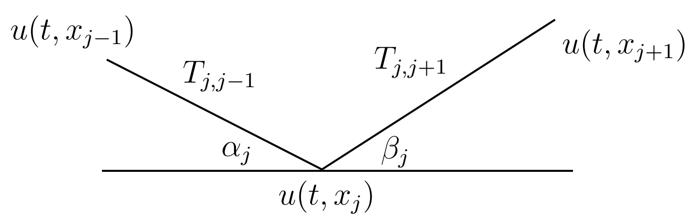
  Let $u(t, x)$ be the vertical displacement.

  Divide the total length ℓ into segments: $\Delta x = \dfrac{\ell}{n}$.

  Then, the mass of each segment is $m = \rho \Delta x$.



- Zoom into a single point:



$$\Delta F(t, x_j) = T_{j,j-1} \sin(\alpha_j) + T_{j,j+1} \sin(\beta_j)$$

Assumptions

1. T is constant; and
2. α_j and β_j are small $s.t.$ $\sin(\alpha_j) \approx \tan(\alpha_j)$ and $\sin(\beta_j) \approx \tan(\beta_j)$.

Then,

$$\sin(\alpha_j) \approx \tan(\alpha_j) = \frac{u(t, x_{j-1}) - u(t, x_j)}{\Delta x}$$

$$\sin(\beta_j) \approx \tan(\beta_j) = \frac{u(t, x_{j+1}) - u(t, x_j)}{\Delta x}.$$

Therefore,

$$\begin{aligned} F(t, x_j) &= T\frac{u(t, x_{j-1}) - u(t, x_j)}{\Delta x} + T\frac{u(t, x_{j+1}) - u(t, x_j)}{\Delta x} \\ &= T\left[\frac{u(t, x_{j-1}) + u(t, x_{j+1}) - 2u(t, x_j)}{\Delta x}\right] \end{aligned}$$

Apply Newton's Law: $F = ma$, we get

$$\begin{aligned} \underbrace{\rho\Delta x}_{\text{mass}} \underbrace{\frac{\partial^2 u}{\partial t^2}(t, x_j)}_{\text{acceleration}} &= \Delta F(t, x_j) \\ \frac{\partial^2 u}{\partial t^2}(t, x_j) &= \frac{\Delta F(t, x_j)}{\rho \Delta x} \\ &= \frac{T}{\rho}\underbrace{\left[\frac{u(t, x_{j-1}) + u(t, x_{j+1}) - 2u(t, x_j)}{\Delta x^2}\right]}_{\text{similar to second-order derivative}} \end{aligned}$$

Take the limit $\Delta x \to 0$, i.e., $n \to +\infty$, we get

$$\lim_{\Delta x \to 0} \frac{u(t, x_{j-1}) + u(t, x_{j+1}) - 2u(t, x_j)}{\Delta x^2} = \frac{\partial^2 u}{\partial x^2}.$$

- Step 3: From discrete back to continuous:

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}$$

So, we get the *1D Wave Equation*:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \underbrace{\dfrac{T}{\rho}}_{=\gamma^2} \dfrac{\partial^2 u}{\partial x^2} = 0 \\ \text{[ICs]} \quad \begin{cases} u(0, x) = u_0(x) \\ \dfrac{\partial u}{\partial t}(0, x) = v_0(x) \end{cases} \\ \text{[BCs]} \quad \begin{cases} u(t, 0) = 0 \\ u(t, \ell) = 0 \end{cases} \end{cases} \tag{1D Wave Equation}$$

□

3.2 D'Alembert's Formula

Theorem 3.2.1 D'Alembert's Formula
The IVP

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \gamma^2 \frac{\partial^2 u}{\partial x^2} = 0 \\ u(0,x) = u_0(x) \\ \frac{\partial u}{\partial t}(0,x) = v_0(x). \end{cases}$$

for $u_0 \in \mathcal{C}^2(\mathbb{R}^2)$, $v \in \mathcal{C}^1(\mathbb{R})$ has the *unique solution*

$$u(t,x) = \frac{1}{2}[u_0(x+\gamma t) + u_0(x-\gamma t)] + \frac{1}{2\gamma}\int_{x-\gamma t}^{x+\gamma t} v_0(\kappa)\,\mathrm{d}\kappa.$$

3.2.1 Proof by Reducing to A System of First Order PDEs

Let's first recall how we can solve a second order ODE using systems. Consider

$$y'' + p(t)y' + q(t)y = 0.$$

Define

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{where } x_1 \coloneqq y \text{ and } x_2 \coloneqq y'.$$

We aim to build $x' = Ax$. The following will work:

$$x' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ -p(t)y' - q(t)y \end{bmatrix} = \begin{bmatrix} x_2 \\ -p(t)x_2 - q(t)x_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We will use a similar method to prove the D'Alembert's formula by reducing the second order PDE into a system of first order PDEs.

Consider the following IVP:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \gamma^2 \frac{\partial^2 u}{\partial x^2} = 0 \\ u(0,x) = u_0(x) \\ \frac{\partial u}{\partial t} = v_0(x). \end{cases}$$

Denote

$$u_1 \coloneqq \frac{\partial u}{\partial x} \quad \text{and} \quad u_2 \coloneqq \frac{\partial u}{\partial t}.$$

Recall from exact equation: Once we know $\frac{\partial u}{\partial x}$ *and* $\frac{\partial u}{\partial t}$*, we can solve for* u*.*

Let's rewrite the PDE:

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) - \gamma^2 \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = 0$$

$$\boxed{\frac{\partial u_2}{\partial t} - \gamma^2 \frac{\partial u_1}{\partial x} = 0}$$

We need another equation, and we obtain this equation from the requirement that u *must be of* $\mathcal{C}^2$. That is,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial u_2}{\partial x} = \frac{\partial u_1}{\partial t}$$

$$\boxed{\frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial x} = 0}$$

So, the system is

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \dfrac{\partial u_2}{\partial x} = 0 \\ \\ \dfrac{\partial u_2}{\partial t} - \gamma^2 \dfrac{\partial u_1}{\partial x} = 0 \\ \\ u_1(0,x) = \dfrac{\partial}{\partial x} u_0(x) \\ u_2(0,x) = v_0(x). \end{cases}$$

Then, in matrix form, we have $\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0$, where $A = \begin{bmatrix} 0 & -1 \\ -\gamma^2 & 0 \end{bmatrix}$. This is exactly the problems we worked on in last section!

From previous work, we know u_1 and u_2:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial t \end{bmatrix}.$$

The remaining job is to find $u(t,x)$:

$$\begin{aligned} u(t,x) &= \int u_1(t,x)\,\mathrm{d}x + \varphi(t) \\ &= \int \frac{\partial u}{\partial x}\,\mathrm{d}x + \varphi(t) \\ &= \int \frac{1}{2\gamma}\left[\gamma \frac{\partial u_0}{\partial x}(x+\gamma t) + v_0(x+\gamma t) + \gamma \frac{\partial u_0}{\partial x}(x-\gamma t) - v_0(x-\gamma t)\right]\mathrm{d}x + \varphi(t) \end{aligned}$$

That is,

$$\begin{aligned}
u(t,x) &= \frac{1}{2\gamma}\left[\gamma \int \frac{\partial u_0}{\partial x}(x+\gamma t) + \frac{\partial u_0}{\partial x}(x-\gamma t)\,\mathrm{d}x + \int v_0(x+\gamma t) - v_0(x-\gamma t)\,\mathrm{d}x\right] + \varphi(t) \\
&= \frac{1}{2}\Big[u_0(x+\gamma t) + u_0(x-\gamma t)\Big] + \frac{1}{2\gamma}\int_{x-\gamma t}^{x+\gamma t} v_0(\kappa)\,\mathrm{d}\kappa + \varphi(t).
\end{aligned}$$

So, the final job is the find $\varphi(t)$: compute $\dfrac{\partial u(t,x)}{\partial t}$ and compare against u_2. [$u_0(x)$ *is a singular variable function.*]

$$\begin{aligned}
\frac{\partial u(t,x)}{\partial t} &= \frac{1}{2}\Big[\gamma u_0'(x+\gamma t) - \gamma u_0'(x-\gamma t)\Big] + \frac{1}{2\gamma}[\gamma v_0(x+\gamma t) - (-\gamma) v_0(x-\gamma t)] + \varphi'(t) \\
&= \frac{1}{2\gamma}\Big[\gamma^2 u_0'(x+\gamma t) - \gamma^2 u_0'(x-\gamma t)\Big] + \frac{1}{2\gamma}[\gamma v_0(x+\gamma t) + \gamma v_0(x-\gamma t)] + \varphi'(t) \\
&= u_2.
\end{aligned}$$

So, it must be $\varphi'(t) = 0$.

Since we are doing definite integrals, $\varphi'(t) = 0 \implies \varphi(t) = 0$. So, we recover the D'Alembert's formula:

$$u(t,x) = \frac{1}{2}\Big[u_0(x+\gamma t) + u_0(x-\gamma t)\Big] + \frac{1}{2\gamma}\int_{x-\gamma t}^{x+\gamma t} v_0(\kappa)\,\mathrm{d}\kappa.$$

3.2.2 Proof by Reducing to Two First Order Linear Conservation Laws

Again, consider the following IVP:

$$\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} - \gamma^2 \dfrac{\partial^2 u}{\partial x^2} = 0 \\
u(0,x) = u_0(x) \\
\dfrac{\partial u}{\partial t}(0,x) = v_0(x).
\end{cases}$$

Apply algebra formulas on operators, we get

$$\left(\frac{\partial}{\partial t} + \gamma \frac{\partial}{\partial x}\right) \underbrace{\left(\frac{\partial}{\partial t} - \gamma \frac{\partial}{\partial x}\right) u}_{=:w} = 0$$

Then, we have two first-order linear conservation laws to solve.

1. $\begin{cases}
\dfrac{\partial w}{\partial t} + \gamma \dfrac{\partial w}{\partial x} = 0 \\
w(0,x) = \dfrac{\partial u}{\partial t}(0,x) - \gamma \dfrac{\partial u}{\partial x}(0,x)
\end{cases}$

Using the method of characteristics, we get

- $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \gamma \\ x(0) = x_0 \end{cases} \implies x(t) = \gamma t + x_0 \implies x_0 = x - \gamma t.$
- $\begin{cases} \dfrac{Dw}{Dt} = 0 \\ w(0, x(0)) = \dfrac{\partial u}{\partial t}(0, x(0)) - \gamma \dfrac{\partial u}{\partial x}(0, x(0)) \end{cases}$

  So,

$$\begin{aligned} w(t, x(t)) &= v_0(x_0) - \gamma \frac{\partial u_0}{\partial x}(x_0) \\ w(t, x) &= v_0(x - \gamma t) - \gamma \frac{\partial u_0}{\partial x}(x - \gamma t). \end{aligned}$$

Recall

$$w = \left(\frac{\partial}{\partial t} - \gamma \frac{\partial}{\partial x} \right) u = \frac{\partial u}{\partial t} - \gamma \frac{\partial u}{\partial x}.$$

So, we have the second first-order linear conservation law:

2. $\begin{cases} \dfrac{\partial w}{\partial t} - \gamma \dfrac{\partial u}{\partial t} = w \\ u(0, x) = u_0(x). \end{cases}$

Again, we use the method of characteristics. The characteristic line is $x(t) = -\gamma t + x_0$. So, $x_0 = x + \gamma t$.

Further,

$$\begin{aligned} \frac{Du}{Dt} = w(t, x(t)) \implies \int_0^t \frac{Du}{Dt}(s, x(s)) \, \mathrm{d}s &= \int_0^t w(s, x(s)) \, \mathrm{d}s \\ u(t, x(t)) - u(0, x(0)) &= \int_0^t w(s, x(s)) \, \mathrm{d}s \\ u(t, x(t)) - u_0(x_0) &= \int_0^t w(s, x(s)) \, \mathrm{d}s \\ &= \int_0^t w(s, -\gamma s + x_0) \, \mathrm{d}s \\ &= \int_0^t w(s, -\gamma s + x + \gamma t) \, \mathrm{d}s \\ &= \int_0^t w(s, x + \gamma(t - s)) \, \mathrm{d}s. \end{aligned}$$

Let's work on the RHS first. Since $w(t,x) = v_0(x-\gamma t) - \gamma \dfrac{\partial u_0}{\partial t}(x-\gamma t)$,

$$\begin{aligned} w(s, x+\gamma(t-s)) &= v_0(x+\gamma(t-s)-\gamma s) - \gamma \frac{\partial u_0}{\partial t}(x+\gamma(t-s)-\gamma s) \\ &= v_0(x+\gamma(t-2s)) - \gamma \frac{\partial u_0}{\partial t}(x+\gamma(t-2s)). \end{aligned}$$

So,

$$\int_0^t w(s, x+\gamma(t-s)) = \int_0^t v_0(x+\gamma(t-2s)) - \gamma \frac{\partial u_0}{\partial t}(x+\gamma(t-2s))\, \mathrm{d}s$$

Let $\kappa = x+\gamma(t-2s)$. Then $\mathrm{d}\kappa = -2\gamma\, \mathrm{d}s \implies \mathrm{d}s = -\dfrac{1}{2\gamma}\, \mathrm{d}\kappa$.

- When $s=0$, $\kappa = x+\gamma t$
- When $s=t$, $\kappa = x+\gamma(t-2t) = x-\gamma t$.

Further, since $\kappa(x) = x+\gamma(t-2s)$, we have

$$\frac{\partial u_0}{\partial x} = \frac{\partial u_0}{\partial \kappa} \cdot \underbrace{\frac{\partial \kappa}{\partial x}}_{=1} \implies \frac{\partial u_0}{\partial x} = \frac{\partial u_0}{\partial \kappa}.$$

Then,

$$\begin{aligned} \int_0^t w(s, x+\gamma(t-s)) &= \int_{x+\gamma t}^{x-\gamma t} v_0(\kappa) - \gamma \frac{\partial u_0}{\partial \kappa}(\kappa) \left(-\frac{1}{2\gamma}\, \mathrm{d}\kappa\right) \\ &= \frac{1}{2\gamma} \int_{x-\gamma t}^{x+\gamma t} v_0(\kappa) - \gamma \frac{\partial u_0}{\partial \kappa}(\kappa)\, \mathrm{d}\kappa \end{aligned}$$

Eventually, we get

$$\int_0^t w(s, x+\gamma(t-s)) = \frac{1}{2\gamma} \int_{x-\gamma t}^{x+\gamma t} v_0(\kappa)\, \mathrm{d}\kappa - \frac{1}{2\cancel{\gamma}} \cancel{\gamma} \Big[u_0(x+\gamma t) - u_0(x-\gamma t) \Big].$$

Hence,

$$\begin{aligned} u(t, x(t)) - u_0(x_0) &= -\frac{1}{2}\Big[u_0(x+\gamma t) - u_0(x-\gamma t)\Big] + \frac{1}{2\gamma}\int_{x-\gamma t}^{x+\gamma t} v_0(\kappa)\, \mathrm{d}\kappa \\ u(t,x) &= u_0(x+\gamma t) - \frac{1}{2}\Big[u_0(x+\gamma t) - u_0(x-\gamma t)\Big] + \frac{1}{2\gamma}\int_{x-\gamma t}^{x+\gamma t} v_0(\kappa)\, \mathrm{d}\kappa \\ &= \frac{1}{2}\Big[u_0(x+\gamma t) + u_0(x-\gamma t)\Big] + \frac{1}{2\gamma}\int_{x-\gamma t}^{x+\gamma t} v_0(\kappa)\, \mathrm{d}\kappa \end{aligned}$$

3.2.3 Applying D'Alembert's Formula

Example 3.2.2 Motion of a Simple Square Wave

$$\frac{\partial^2 u}{\partial t^2} - \gamma^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{with } u(0,x) = \begin{cases} 1 \text{ for } |x| \le 1 \\ 0 \text{ o/w,} \end{cases} \quad , \quad \frac{\partial u}{\partial t}(0,x) = 0.$$

Find the solution using D'Alembert formula.

Solution 1.

$$u_0(x) = \begin{cases} 1, |x| \le 1 \\ 0, \text{ o/w} \end{cases} \quad , \quad v_0(x) = 0.$$

[*Even though* $u_0(x) \notin \mathcal{C}^2(\mathbb{R})$*, we can still apply D'Alembert's formula.*]

By D'alembert's formula, we have

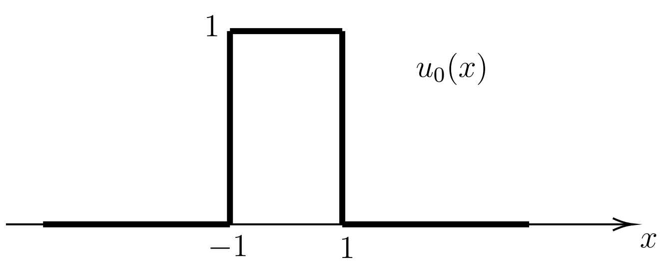
$$u(t,x) = \frac{1}{2}\Big[u_0(x+\gamma t) + u_0(x-\gamma t)\Big] + 0 = \frac{1}{2}\Big[u_0(x+\gamma t) + u_0(x-\gamma t)\Big].$$

To visualize the solution, note that

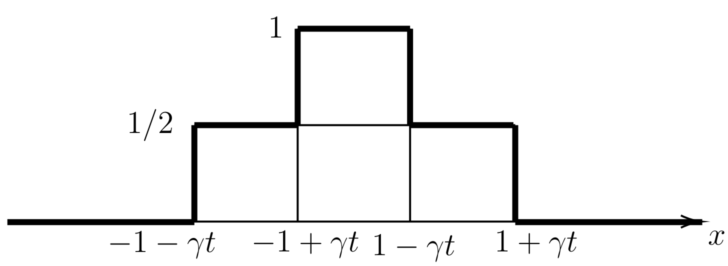
$$u_0(x+\gamma t) = \begin{cases} 1 & 1-\gamma t \le x \le x - \gamma t \\ 0 & \text{o/w} \end{cases} \quad \text{and} \quad u_0(x-\gamma t) = \begin{cases} 1 & 1-\gamma t \le x \le 1+\gamma t \\ 0 & \text{o/w} \end{cases}$$

Then, the snapshots of solutions are

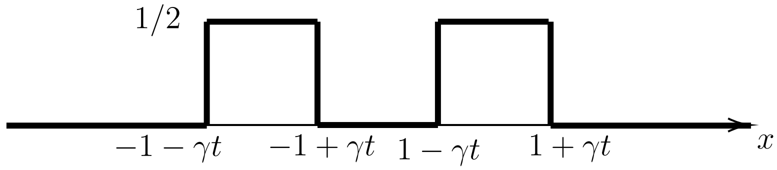
- $t = 0$:



- $1 + \gamma t < 1 - \gamma t \implies 2\gamma t < 2 = \implies t < \frac{1}{\gamma}$:



- $-1+\gamma t > 1-\gamma t \implies t > \dfrac{1}{\gamma}$:



□

Example 3.2.3

Find the solution using D'Alembert's formula:

$$\frac{\partial^2 u}{\partial t^2} - \gamma^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{with } u(0,x) = 0, \quad \frac{\partial u}{\partial t}(0,x) = \begin{cases} 1, & \text{for } |x| \le 1 \\ 0, & \text{o/w}. \end{cases}$$

Solution 2.

From D'Alembert's formula, we know

$$u(t,x) = \frac{1}{2\gamma} \int_{x-\gamma t}^{x+\gamma t} v_0(\kappa)\, \mathrm{d}\kappa.$$

Now, let's discuss different cases (i.e., relationship among -1, 1, $x-\gamma t$, and $x+\gamma t$ to solve the integral.

- No overlapping:
  1. $x-\gamma t < x+\gamma t < -1 < 1$: $u_1 = 0$
  2. $-1 < 1 < x-\gamma t < x+\gamma t$: $u_2 = 0$

- Partial overlapping:
  1. $x-\gamma t < -1 < x+\gamma t < 1$:
  $$u_3 = \frac{1}{2\gamma} \int_{-1}^{x+\gamma t} 1\, \mathrm{d}\kappa = \frac{x+\gamma t+1}{2\gamma}.$$
  2. $-1 < x-\gamma t < 1 < x+\gamma t$:
  $$u_4 = \frac{1}{2\gamma} \int_{x-\gamma t}^{1} 1\, \mathrm{d}\kappa = \frac{1-x+\gamma t}{2\gamma}.$$

- Total overlapping:

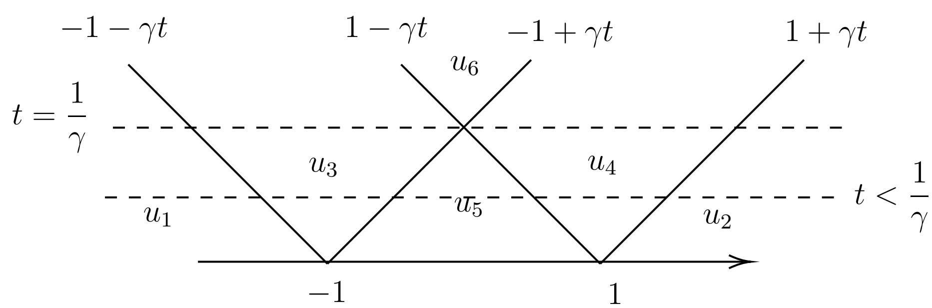
1. $-1 < x - \gamma t < x + \gamma t < 1$:

$$u_5 = \frac{1}{2\gamma}\int_{x-\gamma t}^{x+\gamma t} 1 \, \mathrm{d}\kappa = \frac{x + \gamma t - x + \gamma t}{2\gamma} = t.$$

2. $x - \gamma t < -1 < 1 < x + \gamma t$:

$$u_6 = \frac{1}{2\gamma}\int_{-1}^{1} 1 \, \mathrm{d}\kappa = \frac{1+1}{2\gamma} = \frac{1}{\gamma}.$$

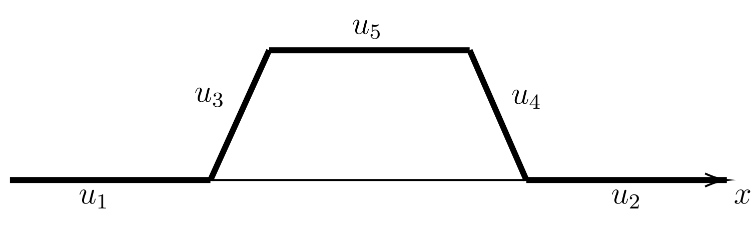
Combining all the situations, we can draw the solution:



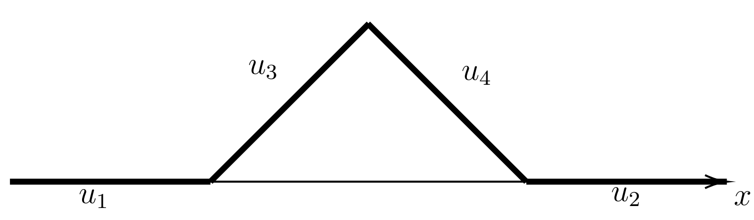
Let's look at a more specific example. Suppose $t = \frac{1}{2\gamma}$. Then,

$$u_3 = \frac{x + \gamma\left(\frac{1}{\gamma}\right) + 1}{2\gamma} = \frac{x + \frac{3}{2}}{2\gamma}, \quad u_5 = t = \frac{1}{2\gamma}, \quad u_6 = \frac{1 - x - \gamma\left(\frac{1}{2\gamma}\right)}{2\gamma} = \frac{-x + \frac{3}{2}}{2\gamma}.$$

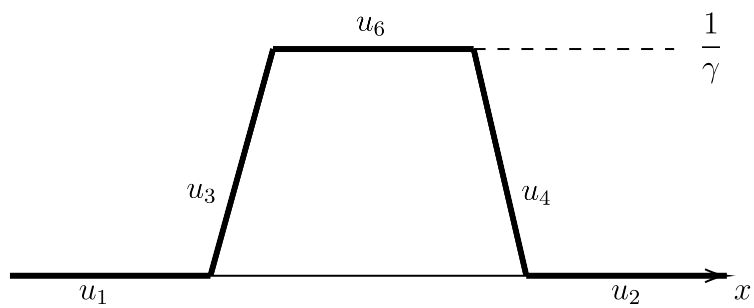
Visualizing it, we get



Further, when $t = \frac{1}{\gamma}$, we get



When $t > \dfrac{1}{\gamma}$, we have



$\square$

4 Heat Equation

4.1 Introduction

$$\frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = f$$

is a *one-dimensional diffusion equation*, where α is called the diffusion coefficient.

To derive this equation, let's consider *Heat Conduction* on a homogenous, isomorphic bar. Let ρ be the constant mass density. Assume $e = e(t, x)$ models the thermal energy per unit mass, then the total quantity of thermal energy is given by

$$\int_a^b e\rho \,\mathrm{d}x.$$

Hence, the change of thermal energy is

$$\frac{\mathrm{d}d}{\mathrm{d}t} \int_a^b e\rho \,\mathrm{d}x = \int_a^b \frac{\partial e}{\partial t} \rho \,\mathrm{d}x.$$

Let $q = q(t, x)$ model the heat flux, then the heat flux change is given by

$$qa - qb = -\int_a^b \frac{\partial q}{\partial x} \,\mathrm{d}x.$$

So, by *Law of Conservation of Energy*,

$$\int_a^b \frac{\partial e}{\partial t} \rho \,\mathrm{d}x = -\int_a^b \frac{\partial q}{\partial x} \,\mathrm{d}x$$
$$\int_a^b \frac{\partial e}{\partial t} \rho + \frac{\partial q}{\partial x} \,\mathrm{d}x = 0$$
$$\Longrightarrow \boxed{\frac{\partial e}{\partial t} \rho + \frac{\partial q}{\partial x} = 0}$$

By *Fourier Law of Heat Conduction*, if $u = u(t, x)$ is the absolute temperature and $k > 0$ is the thermal conductivity, we have

$$q = -k \frac{\partial u}{\partial x}.$$

Moreover, if c is the specific heat of the material, then

$$e = cu.$$

Hence, we get

$$\frac{\partial u}{\partial t} - \frac{k}{c\rho} \frac{\partial^2 u}{\partial x^2} = 0,$$

where $\dfrac{k}{c\rho}$ is called the thermal diffusivity.

We also prescribe initial condition and boundary conditions:

- Initial condition: $u(0,x) = u_0(x)$ is the initial temperature profile, and
- Boundary conditions:
  1. Dirichlet BCs:
  $$\begin{cases} u(t,0) = h_0(t) \\ u(t,1) = h_1(t) \end{cases}$$
  2. Neumann BCs:
  $$\begin{cases} \dfrac{\partial u}{\partial x}(t,0) = h_0(t) \\ \\ \dfrac{\partial u}{\partial x}(t,1) = h_1(t) \end{cases}$$
  3. Robin BCs:
  $$\begin{cases} \dfrac{\partial u}{\partial x}(t,0) + \alpha u(t,0) = h_0(t) \\ \\ \dfrac{\partial u}{\partial x}(t,1) + \beta u(t,1) = h_1(t) \end{cases}$$

Definition 4.1.1 (Heat Equation).

$$\begin{array}{rll} \text{[PDE]} & \dfrac{\partial u}{\partial t} - \alpha^2 \dfrac{\partial^2 u}{\partial x^2} = 0 & \text{with } 0 < x < 1, t > 0 \\ \text{[IC]} & u(0,x) = u_0(x) & \text{with } 0 < x < 1 \\ \text{[BCs]} & \begin{cases} \text{Dirichlet} \\ \text{Nuemann} \\ \text{Robin} \end{cases} & \text{with } t > 0 \end{array}$$

4.2 Separation of Variables

Given the IBVP

$$\begin{aligned} &\text{[PDE]} \quad \frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 && \text{with } 0 < x < 1, t > 0 \\ &\text{[BCs]} \quad \begin{cases} u(t,0) = 0 \\ u(t,1) = 0 \end{cases} && \text{with } t > 0 \\ &\text{[IC]} \quad u(0,x) = \sin(n\pi x) && \text{with } 0 < x < 1, n \text{ integer} \end{aligned}$$

We can find solutions to the PDE in the form $u(t,x) = T(t)X(x)$ by transforming the PDE to ODEs. [*This BC is called the* Dirichlet homogenous BCs.]

Assume $u(t,x) = T(t)X(x)$. Then,

$$\begin{aligned} \frac{\partial u}{\partial t} &= T'(t)X(x) \\ \frac{\partial u}{\partial x} &= T(t)X'(x) \implies \frac{\partial^2 u}{\partial x^2} = T(t)X''(x). \end{aligned}$$

Then, the PDE becomes

$$T'(t)X(x) - \alpha^2 T(t)X''(x) = 0.$$

Note that $u(t,x) = 0$ is a trivial solution, and we are not interested in this trivial solution. Assume $u(t,x) \neq 0$. Divide both sides by $\alpha^2 u(t,x)$:

$$\begin{aligned} \frac{T'(t)X(x)}{\alpha^2 u(t,x)} - \frac{\alpha^2 T(t)X''(x)}{\alpha^2 u(t,x)} &= 0 \\ \frac{T'(t)\cancel{X(x)}}{\alpha^2 T(t)\cancel{X(x)}} - \frac{\cancel{\alpha^2 T(t)}X''(x)}{\cancel{\alpha^2 T(t)}X(x)} &= 0 \\ \frac{T'(t)}{\alpha^2 T(t)} - \frac{X''(x)}{X(x)} &= 0 \\ \frac{T'(t)}{\alpha^2 T(t)} &= \frac{X''(x)}{X(x)}. \end{aligned}$$

The only possibility of a function of t equals a function of x is that both of them equal to a constant, say k. So, we have

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = k.$$

That is, we have two ODEs:

- $\frac{T'(t)}{\alpha^2 T(t)} = k \implies T'(t) = k\alpha^2 T(t) \implies T'(t) - k\alpha^2 T(t) = 0$. This is a first-order linear homogenous ODE, and we have the general solution formula:

$$T(T) = Ce^{k\alpha^2 t}.$$

[*Recall that* $u(t,x) = T(t)X(x)$ *is the temperature. So, we expect* $T(T) = Ce^{k\alpha^2 t} \to 0$ *when* $t \to \infty$. *Since* $\alpha^2 > 0$ *and* $t > 0$, *we need* $k < 0$. *We will also see this condition later in the discussion of cases.*]

- $\dfrac{X''(x)}{X(x)} = k \implies X''(x) = kX(x) \implies X''(x) - kX(x) = 0$. This is a second-order, linear, homogenous, constant coefficients ODE. We will solve using characteristic polynomial:

$$p(r) = r^2 - k \overset{\text{set}}{=} 0.$$

Case I : $k > 0$. Since $r^2 = k$, $r_{1,2} = \pm\sqrt{k}$. Then, the Fundamental Set of Solutions (FSS) is $\left\{e^{\sqrt{k}x}, e^{-\sqrt{k}x}\right\}$. Imposing BCs, we have

$$X(0) = 0 \quad \text{and} \quad X(1) = 0.$$

Let $X(t) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}$. Then,

$$\begin{cases} X(0) = A + B = 0 \\ X(1) = Ae^{\sqrt{k}} + Be^{-\sqrt{k}} = 0 \end{cases}$$

Write it in the matrix form:

$$\begin{bmatrix} 1 & 1 \\ e^{\sqrt{k}} & e^{-\sqrt{k}} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

[*Recall:* $A\vec{\mathbf{x}} = \vec{\mathbf{0}} \begin{cases} \det(A) \neq 0 \implies \textit{unique solution: } \vec{\mathbf{x}} = \vec{\mathbf{0}} \\ \det(A) = 0 \implies \textit{infinitely many solutions} \end{cases}$] Since

$$\det \begin{bmatrix} 1 & 1 \\ e^{\sqrt{k}} & e^{-\sqrt{k}} \end{bmatrix} = e^{-\sqrt{k}} - e^{\sqrt{k}} \neq 0 \quad \text{when } k > 0,$$

we have a unique solution to the system

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence, $X(t) = 0 \cdot e^{\sqrt{k}x} + 0 \cdot e^{-\sqrt{k}x} = 0$ and $u(t,x) = T(t)X(x) = 0$. This is not interesting!

Case II : $k = 0$. Then, we have $X''(x) = 0$, which implies $X(x)$ is a linear function.

Assume $X(x) = Ax + B$. Imposing BCs, we have

$$\begin{cases} X(0) = B = 0 \\ X(1) = A + B = 0 \end{cases} \implies \begin{cases} A = 0 \\ B = 0 \end{cases}$$

Then, $X(x) = 0 \implies u(t,x) = T(t)X(x) = 0$. Not interesting!

$\boxed{\text{Case III}}$: $k < 0$. Let's define $k = -\lambda^2$. So,

$$\begin{aligned} X''(x) - (-\lambda^2)X(x) &= 0 \\ X''(x) + \lambda^2 X(x) &= 0. \end{aligned}$$

Solving the characteristic polynomial, we need

$$\begin{aligned} p(r) = r^2 + \lambda^2 &\overset{\text{set}}{=} 0 \\ r^2 &= -\lambda^2 \\ r_{1,2} &= \pm \mathrm{i}\lambda. \end{aligned}$$

So, FSS $= \left\{e^{\mathrm{i}\lambda x}, e^{-\mathrm{i}\lambda x}\right\}$. By Euler's formula, we have

$$\begin{aligned} e^{\mathrm{i}\lambda x} &= \cos(\lambda x) + \mathrm{i}\sin(\lambda x) \\ e^{-\mathrm{i}\lambda x} &= \cos(\lambda x) - \mathrm{i}\sin(\lambda x). \end{aligned}$$

[*Recall:* $z = a + \mathrm{i}b$ *and its conjugate* $\overline{z} = a - \mathrm{i}b$. *Then,*

$$\begin{aligned} \frac{z + \overline{z}}{2} &= \frac{a + \mathrm{i}b + a - \mathrm{i}b}{2} = \frac{2a}{2} = a = \mathrm{Re}(z) \in \mathbb{R} \\ \frac{z - \overline{z}}{2\mathrm{i}} &= \frac{a + \mathrm{i}b - a + \mathrm{i}b}{2\mathrm{i}} = \frac{2\mathrm{i}b}{2\mathrm{i}} = b = \mathrm{Im}(z) \in \mathbb{R}. \end{aligned}$$

$\mathrm{Re}(z)$ *and* $\mathrm{Im}(z)$ *are linear combinations of* z *and* $\overline{z}$. *By Principle of Superposition, if* z *and* $\overline{z}$ *are solutions, so do* $\mathrm{Re}(z)$ *and* $\mathrm{Im}(z)$.]

So, by Principle of Superposition, $\mathrm{Re}\left(e^{\mathrm{i}\lambda x}\right)$ and $\mathrm{Im}(e^{\mathrm{i}\lambda x})$ are solutions to the ODE:

$$\begin{aligned} X_1(x) &= \mathrm{Im}\left(e^{\mathrm{i}\lambda x}\right) = \sin(\lambda x) \\ X_2(x) &= \mathrm{Re}\left(e^{\mathrm{i}\lambda x}\right) = \cos(\lambda x). \end{aligned}$$

So, $X(x) = AX_1(x) + BX_2(x) = A\sin(\lambda x) + B\cos(\lambda x)$.

Imposing BCs:

$$\begin{cases} X(0) = A\sin(\lambda \cdot 0) + B\cos(\lambda \cdot 0) = 0 \\ X(1) = A\sin(\lambda \cdot 1) + B\cos(\lambda \cdot 1) = 0 \end{cases}$$

$$\implies \begin{cases} B = 0 \\ A\sin(\lambda) + B\cos(\lambda) = 0 \end{cases} \implies \begin{cases} A\sin(\lambda) = 0 \\ B = 0 \end{cases}$$

Then, we have two possibilities:

1. $A = 0$. Then, we are back into cases of $X(x) = 0$ and $u(t,x) = 0$. Not interesting!
2. $\sin(\lambda) = 0$. Then, $\lambda = \gamma\pi$, where $\gamma = 1, 2, \ldots$. Then, $X(x) = A\sin(\gamma\pi x)$. This is the interesting case.

Hence, the solution is

$$u(t,x) = T(t)X(x) = Ce^{k\alpha^2 t}A\sin(\gamma\pi x).$$

Recall that $k = -\lambda^2$ and $\lambda = \gamma\pi$. So, $k = -(\gamma\pi)^2$. Then,

$$u(t,x) = CAe^{-(\gamma\pi)^2\alpha^2 t}\sin(\gamma\pi x).$$

Finally, let's impose IC:

$$\begin{aligned} u(0,x) &= \sin(n\pi x) \\ CAe^0\sin(\gamma\pi x) &= \sin(n\pi x) \\ CA\sin(\gamma\pi x) &= \sin(n\pi x) \end{aligned}$$

So, $CA = 1$ and $\gamma = n$. Therefore, our solution of the IBVP is

$$\boxed{u(t,x) = e^{-(n\pi)^2\alpha^2 t}\sin(n\pi x)}.$$

Example 4.2.1 Principle of Superposition

Given the IBVP

[PDE]	$\dfrac{\partial u}{\partial t} - \alpha^2\dfrac{\partial^2 u}{\partial x^2} = 0$	with $0 < x < 1, t > 0$
[BCs]	$\begin{cases} u(t,0) = 0 \\ u(t,1) = 0 \end{cases}$	with $t > 0$
[IC]	$u(0,x) = 3\sin(7\pi x) + 12\sin(10\pi x)$	with $0 < x < 1, n$ integer

Solution 1.
The solution is

$$u(t,x) = CAe^{-(\gamma\pi)^2\alpha^2 t}\sin(\gamma\pi x).$$

Use the principle of superposition, we can split the ICs into 2:

$$\begin{aligned} u_1(0,x) &= 3\sin(7\pi x) \\ u_2(0,x) &= 12\sin(10\pi x). \end{aligned}$$

Then, $u(t,x) = u_1(t,x) + u_2(t,x)$. So,

$$\begin{aligned} u_1(0,x) = CAe^0\sin(\gamma\pi x) &= 3\sin(7\pi x) \\ CA\sin(\gamma\pi x) &= 3\sin(7\pi x) \\ \implies CA = 3, &\quad \gamma = 7 \end{aligned}$$

So, $u_1(t,x) = 3e^{-(7\pi)^2\alpha^2 t}\sin(7\pi x)$. Similarly, $u_2(t,x) = 12e^{-(10\pi)^2\alpha^2 t}\sin(10\pi x)$. So, the solution to the IBVP is

$$\begin{aligned} u(t,x) &= u_1(t,x) + u_2(t,x) \\ &= 3e^{-(7\pi)^2\alpha^2 t}\sin(7\pi x) + 12e^{-(10\pi)^2\alpha^2 t}\sin(10\pi x). \end{aligned}$$

□

4.3 Fourier Series

We can represent a periodic function $f = f(x)$ with period of 2π (i.e., $f(x) = f(x+2\pi)$) in terms of weighted sums of sine and cosine functions:

$$\begin{aligned} f(x) = B_0 \overbrace{\cos(0\cdot x)}^{=1} &+ B_1\cos(x) + B_2\cos(2x) + \cdots + B_N\cos(Nx) + \cdots \\ + A_0 \underbrace{\sin(0\cdot x)}_{=0} &+ A_1\sin(x) + A_2\sin(2x) + \cdots + A_N\sin(Nx) + \cdots \end{aligned}$$

Theorem 4.3.1 Summary of Integrals of Sine and Cosine Functions
Let's consider the interval $[-\pi,\pi]$.

- $\displaystyle\int_{-\pi}^{\pi}\sin(mx)\,\mathrm{d}x = 0.$

- $\displaystyle\int_{-\pi}^{\pi} \cos(mx)\,\mathrm{d}x = 0$.
- $\displaystyle\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,\mathrm{d}x = 0$.
- $\displaystyle\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = 0$ if $m \neq n$.
- $\displaystyle\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = \pi$ if $m = n$.
- $\displaystyle\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x = 0$ if $m \neq n$.
- $\displaystyle\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x = \pi$ if $m = n$.

Suppose we want to find B_1:

$$\begin{aligned} f(x)\cos(x) = B_0\cos(x) + B_1\cos(x)\cos(x) + \cdots + B_N\cos(Nx)\cos(x) + \cdots \\ + A_1\sin(x)\cos(x) + \cdots + A_N\sin(Nx)\cos(x) + \cdots \end{aligned}$$

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\cos(x)\,\mathrm{d}x = B_0 \overbrace{\int_{-\pi}^{\pi}\cos(x)\,\mathrm{d}x}^{=0} + B_1 \overbrace{\int_{-\pi}^{\pi}\cos(x)\cos(x)\,\mathrm{d}x}^{=\pi} + \cdots + B_N \overbrace{\int_{-\pi}^{\pi}\cos(Nx)\cos(x)\,\mathrm{d}x}^{=0} + \cdots \\ + A_1 \underbrace{\int_{-\pi}^{\pi}\sin(x)\cos(x)\,\mathrm{d}x}_{=0} + \cdots + A_N \underbrace{\int_{-\pi}^{\pi}\sin(Nx)\cos(x)\,\mathrm{d}x}_{=0} \end{aligned}$$

So,

$$\int_{-\pi}^{\pi} f(x)\cos(x)\,\mathrm{d}x = \pi B_1 \implies B_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(x)\,\mathrm{d}x$$

In general,

$$\boxed{B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,\mathrm{d}x}.$$

To find B_0:

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = B_0 \overbrace{\int_{-\pi}^{\pi}\mathrm{d}x}^{2\pi} + B_1 \overbrace{\int_{-\pi}^{\pi}\cos(x)\,\mathrm{d}x}^{=0} + \cdots + B_N \underbrace{\int_{-\pi}^{\pi}\cos(Nx)\,\mathrm{d}x}^{=0} + \cdots \\ + A_1 \underbrace{\int_{-\pi}^{\pi}\sin(x)\,\mathrm{d}x}_{=0} + \cdots + A_N \underbrace{\int_{-\pi}^{\pi}\sin(Nx)\,\mathrm{d}x}_{=0} + \cdots \end{aligned}$$

So, we have

$$\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = 2\pi B_0 \implies \boxed{B_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x}.$$

Similarly,

$$\boxed{A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,\mathrm{d}x}.$$

Theorem 4.3.2
Denote

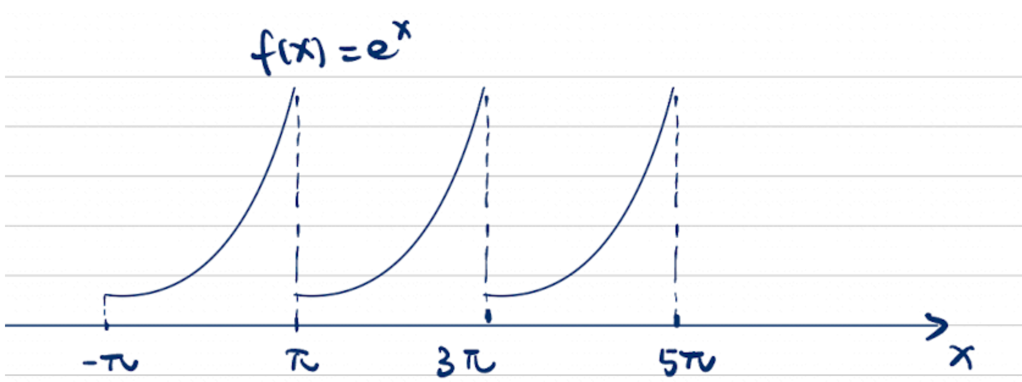
$$p_N(x) = B_0 + \sum_{k=1}^{N} B_k\cos(kx) + A_k\sin(kx).$$

If $f(x)$ has a period of 2π (i.e., $f(x) = f(x+2\pi)$) and is integrable on $[-\pi, pi]$, then

$$\lim_{N\to+\infty} p_N(x) = f(x).$$

Example 4.3.3
Consider the function $f = f(x) = e^x$ on $(-\pi, \pi)$, replicated to be a periodic function on $\mathbb{R}$ of period 2π. Find its Fourier expansion.

Solution 1.



This function is periodic with a period of 2π.

$$B_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x\,\mathrm{d}x = \frac{e^\pi - e^{-\pi}}{2\pi}$$
$$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\cos(kx)\,\mathrm{d}x = \frac{e^\pi - e^{-\pi}}{\pi(k^2+1)}(-1)^k$$
$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\sin(kx)\,\mathrm{d}x = \frac{e^\pi - e^{-\pi}}{\pi(k^2+1)}k(-1)^{k+1}$$

Recall: hyperbolic sine:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}.$$

Then,

$$p_N(x) = \frac{2\sinh(\pi)}{\pi}\left[\frac{1}{2} + \sum_{k=1}^{N} \frac{k(-1)^{k+1}\sin(kx) + (-1)^k\cos(kx)}{k^2+1}.\right]$$

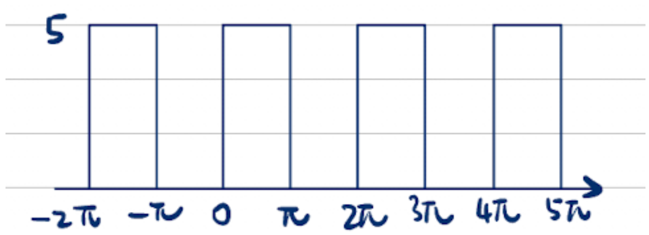
□

Example 4.3.4

Write the following square wave in terms of Fourier Series:

$$f(x) = \begin{cases} 5 & \text{if } 2k\pi < x < (2k+1)\pi \\ 0 & \text{if } (2k+1)\pi < x < (2k+2)\pi \end{cases}$$

Solution 2.



$$B_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{5}{2}$$

$$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,\mathrm{d}x = 0$$

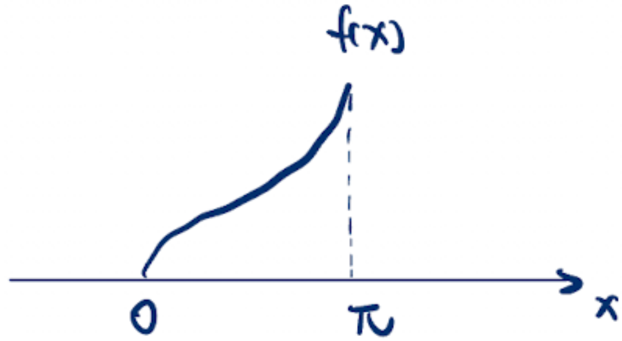
$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,\mathrm{d}x = \begin{cases} 0, & \text{if } k \text{ is even} \\ \dfrac{10}{k\pi} & \text{if } k \text{ is odd.} \end{cases}$$

So,

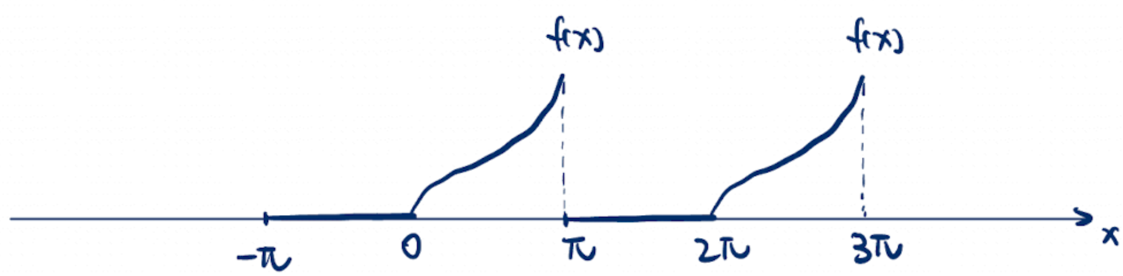
$$\begin{aligned} p_N &= \frac{5}{2} + \sum_{k=0}^{\lfloor N/2 \rfloor} \frac{10}{(2k+1)\pi}\sin((k+1)x) \\ &= \frac{5}{2} + \frac{10}{\pi}\sin(x) + \frac{10}{3\pi}\sin(3x) + \cdots \end{aligned}$$

□

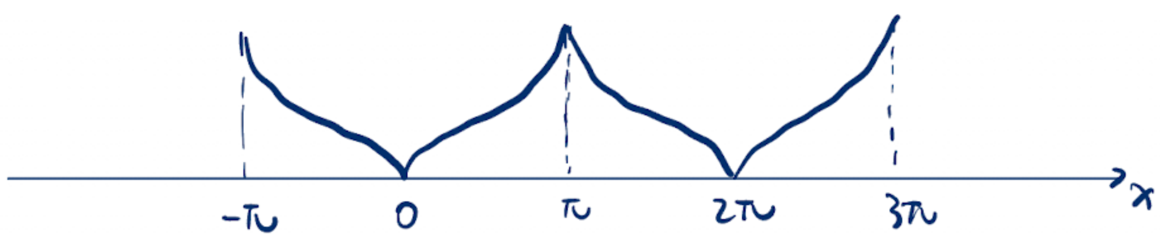
How to make $f(x)$ periodic with a period of 2π?



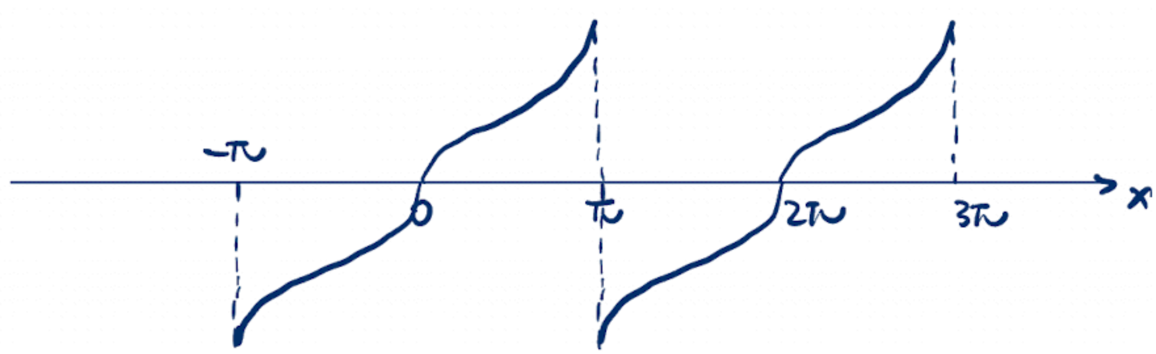
- The zero option: Let $f(x) = 0$ for $x \in [-\pi, 0]$.



- Even function: reflect $w.r.t.$ y-axis.



- Odd function: reflect $w.r.t.$ x-axis.



Properties

- $\displaystyle\int_{-\pi}^{\pi} \mathrm{odd}(x)\,\mathrm{d}x = 0$
- $\displaystyle\int_{-\pi}^{\pi} \mathrm{even}(x)\,\mathrm{d}x = 2\int_{0}^{\pi} \mathrm{even}(x)\,\mathrm{d}x$
- $\displaystyle\int_{-\pi}^{\pi} \underbrace{\mathrm{even}(x)\,\mathrm{odd}(x)}_{\mathrm{odd}}\,\mathrm{d}x = 0$

General Solution of Heat Equation with Dirichlet BCs

$$u(t,x) = \sum_{k=1}^{\infty} e^{-(k\pi)^2\alpha^2 t}[A_k \sin(k\pi x)].$$

So, we don't want any B_0 or B_k terms.

Take $f(x) = \mathrm{odd}(x)$. Then,

$$\begin{aligned}
B_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{odd}(x)\,\mathrm{d}x = 0 \\
B_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{odd}(x)\underbrace{\cos(kx)}_{\mathrm{even}}\,\mathrm{d}x = 0 \\
A_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} \mathrm{odd}(x)\sin(kx)\,\mathrm{d}x
\end{aligned}$$

At $t = 0$, compare $u(t,x)$ against the IC $u_0(x)$:

$$u(0,x) = \sum_{k=1}^{\infty} e^0 A_k \sin(k\pi x) = \sum_{k=1}^{\infty} A_k \sin(k\pi x) = u_0(x).$$

One more step: Change of variable – map from $[0,\pi] \to [0,1]$.

Let $\xi = \pi x$. Then, $x = 0 \implies \xi = 0$ and $x = 1 \implies \xi = \pi$. Also, $\mathrm{d}\xi = \pi \mathrm{d}x$.

We are working on $u_0(\xi)$ because the integrals were defined on $[0,\pi]$. Convert from ξ back to x, we get

$$\begin{aligned}
A_k &= \frac{2}{\pi}\int_0^{\pi} \widetilde{u}_0(\xi)\sin(k\xi)\,\mathrm{d}\xi && [\widetilde{u}_0(\xi) = u_0 \textit{ in terms of } \xi.] \\
&= \frac{2}{\cancel{\pi}}\int_0^{1} u_0(x)\sin(k\pi)\cancel{\pi}\,\mathrm{d}x && [\textit{Change of Variable}]
\end{aligned}$$

$$\boxed{A_k = 2\int_0^1 u_0(x)\sin(k\pi x)\,\mathrm{d}x}.$$

4.3.1 Worked Examples

Example 4.3.5

Solve the IBVP

$$\begin{aligned}
&\text{[PDE]} \quad \frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 && \text{with } 0 < x < 1, t > 0 \\
&\text{[BCs]} \quad \begin{cases} u(t,0) = 0 \\ u(t,1) = 0 \end{cases} && \text{with } t > 0 \\
&\text{[IC]} \quad u(0,x) = e^x && \text{with } 0 < x < 1
\end{aligned}$$

Solution 3.

By separation of variables ($(t,x) = T(t)X(x)$) and by imposing the Dirichlet boundary condition, we can write the solution in the form

$$u(t,x) = \sum_{k=1}^{\infty} e^{-(k\pi)^2\alpha^2 t}[A_k \sin(k\pi x)].$$

By imposing IC, we can find the coefficients A_k:

$$u(0,x) = \sum_{k=1}^{\infty} A_k \sin(k\pi x) = e^x.$$

By Fourier Series of e^x:

$$\begin{aligned}
A_k &= 2\int_0^1 e^x \sin(k\pi x)\,\mathrm{d}x \\
\int e^x \sin(k\pi x))\,\mathrm{d}x &= e^x \sin(k\pi x) - k\pi \int e^x \cos(k\pi x)\,\mathrm{d}x \\
&= e^x \sin(k\pi x) - k\pi \left[e^x \cos(k\pi x) + k\pi \int e^x \sin(k\pi x)\,\mathrm{d}x\right] \\
&= e^x \sin(k\pi x) - k\pi e^x \cos(k\pi x) - (k\pi)^2 \int e^x \sin(k\pi x)\,\mathrm{d}x \\
\left(1 + (k\pi)^2\right) \int e^x \sin(k\pi x)\,\mathrm{d}x &= e^x \sin(k\pi x) - k\pi e^x \cos(k\pi x) \\
\int e^x \sin(k\pi x)\,\mathrm{d}x &= \frac{1}{1 + (k\pi)^2} e^x \sin(k\pi x) - k\pi e^x \cos(k\pi x)
\end{aligned}$$

So,

$$\int_0^1 e^x \sin(k\pi x)\,\mathrm{d}x = \frac{1}{1+(k\pi)^2}[-k\pi e\cos(k\pi) + k\pi]$$
$$= \frac{k\pi}{1+(k\pi)^2}[1 - e\cos(k\pi)] = \begin{cases} (1-e)\dfrac{k\pi}{1+(k\pi)^2} & \text{if } k \text{ even} \\ (1+e)\dfrac{k\pi}{1+(k\pi)^2} & \text{if } k \text{ odd} \end{cases}$$

□

Example 4.3.6

Solve the IBVP

$$\text{[PDE]} \quad \frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{with } 0 < x < 1, t > 0$$
$$\text{[BCs]} \quad \begin{cases} u(t,0) = 0 \\ u(t,1) = 0 \end{cases} \qquad \text{with } t > 0$$
$$\text{[IC]} \quad u(0,x) = \begin{cases} 0 & \text{with } x \le 1/2 \\ 1 & \text{with } x > 1/2 \end{cases} \qquad \text{with } 0 < x < 1$$

Solution 4.

General solution of heat equation with Dirichlet BCs:

$$u(t,x) = \sum_{k=1}^{\infty} e^{-(k\pi)^2\alpha^2 t} A_k \sin(k\pi x),$$

and

$$\begin{aligned} A_k &= 2\int_0^1 u_0(x)\sin(k\pi x)\,\mathrm{d}x \\ &= 2\int_0^{1/2} \underbrace{u_0(x)}_{=0}\sin(k\pi x)\,\mathrm{d}x + 2\int_{1/2}^1 \underbrace{u_0(x)}_{=1}\sin(k\pi x)\,\mathrm{d}x \\ &= 2\int_{1/2}^1 \sin(k\pi x)\,\mathrm{d}x \\ &= -\frac{2}{k\pi}[\cos(k\pi x)]_{1/2}^1 \\ &= -\frac{2}{k\pi}\left[\cos(k\pi) - \cos\left(k\frac{\pi}{2}\right)\right]. \end{aligned}$$

Since

$$\cos(k\pi) = \begin{cases} 1 & k \text{ is even} \\ -1 & k \text{ is odd} \end{cases} \quad \text{and} \quad \cos\left(k\frac{\pi}{2}\right) = \begin{cases} 1 & k \text{ is multiple of } 4 \\ 0 & k \text{ is odd} \\ -1 & k \text{ is even but not multiple of } 4 \end{cases},$$

we have

$$A_k = \begin{cases} -\frac{2}{k\pi}(-1) = \frac{2}{k\pi} & k \text{ is odd} \\ -\frac{2}{k\pi}(0) = 0 & 4 \mid k \\ -\frac{2}{k\pi}(1+1) = -\frac{4}{k\pi} & k \text{ is even but } 4 \nmid k. \end{cases}$$

□

Example 4.3.7 Lifting Function

Solve the IBVP

$$\text{[PDE]} \quad \frac{\partial u}{\partial t} - \gamma^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{with } 0 < x < 1, t > 0$$

$$\text{[BCs]} \quad \begin{cases} u(t,0) = \alpha \\ u(t,1) = \beta \end{cases} \qquad \text{with } t > 0, \alpha, \beta \in \mathbb{R}$$

$$\text{[IC]} \quad u(0,x) = \sin(n\pi x) \qquad \text{with } 0 < x < 1, n \text{ a given integer}$$

Idea Write $u(t,x)$ as the sum of two functions. One function satisfy the problem with homogeneous Dirichlet condition, and the other function captures whatever is leftover.

$$u(t,x) = \widetilde{u}(t,x) + \ell(t,x),$$

where ℓ is called the lifting function.

Solution 5.

We want $\ell(t,x)$ to satisfy:

$$\ell(t,0) = \alpha \quad \text{and} \quad \ell(t,1) = \beta.$$

Then,

$$u(t,0) = \widetilde{u}(t,0) + \ell(t,0)$$

So,

$$\begin{aligned}\widetilde{u}(t,0) &= u(t,0) - \ell(t,0) = \alpha - \alpha = 0.\\ \widetilde{u}(t,1) &= u(t,1) - \ell(t,1) = \beta - \beta = 0.\end{aligned}$$

Assume ℓ is linear $w.r.t.\ x$:

$$\ell(t,x) = P(t)x + Q(t).$$

Imposing BCs:

$$\begin{aligned}\ell(t,0) &= P(t)\cdot 0 + Q(t) = \alpha \implies Q(t) = \alpha,\\ \ell(t,1) &= P(t)\cdot 1 + Q(t) = \beta \implies P(t) = \beta - \alpha.\end{aligned}$$

So,

$$\ell(t,x) = (\beta - \alpha)x + \alpha.$$

Hence,

$$\begin{aligned}u(t,x) &= \widetilde{u}(t,x) + \ell(t,x)\\ &= \widetilde{u}(t,x) + (\beta-\alpha)x + \alpha.\end{aligned}$$

Rewrite the problem in $\widetilde{u}(t,x)$:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial \widetilde{u}}{\partial t} + \frac{\partial \ell}{\partial t} = \frac{\partial \widetilde{u}}{\partial t}\\ \frac{\partial t}{\partial x} &= \frac{\partial \widetilde{u}}{\partial x} + \frac{\partial \ell}{\partial x} = \frac{\partial \widetilde{u}}{\partial x} + (\beta - \alpha).\\ \implies \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 \widetilde{u}}{\partial x^2} + \frac{\partial}{\partial x}(\beta-\alpha) = \frac{\partial^2 \widetilde{u}}{\partial x^2}.\end{aligned}$$

Also,

$$\begin{aligned}u(0,x) &= \widetilde{u}(0,x) + \ell(0,x)\\ \implies \widetilde{u}(0,x) &= u(0,x) - \ell(0,x) = \sin(n\pi x) - (\beta-\alpha)x - \alpha.\end{aligned}$$

So, the problem becomes:

$$\begin{cases}\dfrac{\partial \widetilde{u}}{\partial t} - \gamma^2 \dfrac{\partial^2 \widetilde{u}}{\partial x^2} = 0\\ \widetilde{u}(t,0) = \widetilde{u}(t,1) = 0\\ \widetilde{u}(0,x) = \sin(n\pi x) - (\beta-\alpha)x - \alpha.\end{cases}$$

By separation of variable, imposing Dirichlet BCs, and principle of superposition, we have

$$u(t,x) = \sum_{k=1}^{\infty} e^{-(k\pi)^2\alpha^2 t} A_k \sin(k\pi x),$$

where

$$\begin{aligned} A_k &= 2\int_0^1 \widetilde{u}_0(x)\sin(k\pi x)\,\mathrm{d}x \\ &= 2\int_0^1 [\sin(n\pi x) - (\beta-\alpha)x - \alpha]\sin(k\pi x)\,\mathrm{d}x \end{aligned}$$

So,

$$u(t,x) = \underbrace{\sum_{k=1}^{\infty}\left[e^{-(k\pi)^2\alpha^2 t} A_k \sin(k\pi x)\right]}_{\widetilde{u}(t,x)} + \underbrace{(\beta-\alpha)x+\alpha}_{\ell(t,x)}.$$

□

Example 4.3.8

Solve the IBVP

$$\begin{aligned} &[\text{PDE}] \quad \frac{\partial u}{\partial t} - \alpha^2\frac{\partial^2 u}{\partial x^2} = 0 && \text{with } 0<x<1, t>0 \\ &[\text{BCs}] \quad \begin{cases} u(t,0)=0 \\ u(t,1)=t \end{cases} && \text{with } t>0 \\ &[\text{IC}] \quad u(0,x) = u_0(x) && \text{with } 0<x<1 \end{aligned}$$

Solution 6.

Suppose $u(t,x) = \widetilde{u}(t,x) + \ell(t,x)$, where $\ell(t,0)=0$ and $\ell(t,1)=t$.
Assume $\ell(t,x) = P(t)x + Q(t)$. Imposing BCs, we have

$$\begin{aligned} \ell(t,0) &= P(t)\cdot 0 + Q(t) = 0 \implies Q(t) = 0 \\ \ell(t,1) &= P(t)\cdot 1 + Q(t) = t \implies P(t) = t. \end{aligned}$$

So, $\ell(t,x) = tx \implies u(t,x) = \widetilde{u}(t,x) + tx$. Then,

$$\frac{\partial u}{\partial t} = \frac{\partial \widetilde{u}}{\partial t} + \frac{\partial \ell}{\partial t} = \frac{\partial \widetilde{u}}{\partial t} + x \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 \widetilde{u}}{\partial x^2} + \frac{\partial^2 \ell}{\partial x^2} = \frac{\partial^2 \widetilde{u}}{\partial x^2}$$

Also,

$$u(0,x) = \widetilde{u}(0,x) + 0 \cdot x = u_0(x) \implies \widetilde{u}(0,x) = u_0(x).$$

Then, the problem becomes

$$\frac{\partial \widetilde{u}}{\partial t} + x - \alpha^2 \frac{\partial^2 \widetilde{u}}{\partial x^2} = 0$$

That is,

$$\begin{cases} \dfrac{\partial \widetilde{u}}{\partial t} - \alpha^2 \dfrac{\partial^2 \widetilde{u}}{\partial x^2} = -x \\ \widetilde{u}(t,0) = \widetilde{u}(t,1) = 0 \\ \widetilde{u}(0,x) = u_0(x). \end{cases}$$

This PDE is *non-homogeneous*, so we don't know how to solve it yet... □

Example 4.3.9

Solve the IBVP

$$\text{[PDE]} \quad \frac{\partial u}{\partial t} - \gamma^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{with } 0 < x < 1, t > 0$$

$$\text{[BCs]} \quad \begin{cases} u(t,0) = 1 \\ u(t,1) = e \end{cases} \qquad \text{with } t > 0$$

$$\text{[IC]} \quad u(0,x) = e^x \qquad \text{with } 0 < x < 1$$

Solution 7.

Suppose $u(t,x) = \widetilde{u}(t,x) + \ell(t,x)$, where $\ell(t,0) = 1$ and $\ell(t,1) = e$.
Assume $\ell(t,x) = P(t)x + Q(t)$. Then,

$$\ell(t,0) = P(t) \cdot 0 + Q(t) = 1 \implies Q(t) = 1$$
$$\ell(t,1) = P(t) \cdot 1 + Q(t) = e \implies P(t) = e - 1.$$

So,

$$\ell(t,x) = (e-1)x + 1 \implies u(t,x) = \widetilde{u}(t,x) + (e-1)x + 1.$$

Then,

$$\frac{\partial u}{\partial t} = \frac{\partial \widetilde{u}}{\partial t} + \frac{\partial \ell}{\partial t} = \frac{\partial \widetilde{u}}{\partial t} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 \widetilde{u}}{\partial x^2} + \frac{\partial^2 \ell}{\partial x^2} = \frac{\partial^2 \widetilde{u}}{\partial x^2}.$$

Also,

$$u(0,x) = \widetilde{u}(0,x) + (e-1)x + 1 \implies \widetilde{u}(0,x) = u(0,x) - (e-1)x - 1 = e^x - (e-1)x - 1.$$

So, the problem becomes

$$\begin{cases} \dfrac{\partial \widetilde{u}}{\partial t} - \alpha^2 \dfrac{\partial^2 \widetilde{u}}{\partial x^2} = 0 \\ u(t,0) = u(t,1) = 0 \\ \widetilde{u}(0,x) = e^x - (e-1)x - 1. \end{cases}$$

The solution is

$$\widetilde{u}(t,x) = \sum_{k=1}^{\infty} e^{-(k\pi)^2\alpha^2 t^2} A_k \sin(k\pi x),$$

where

$$A_k = 2\int_0^1 \left(e^x - (e-1)x - 1\right)\sin(k\pi x)\,\mathrm{d}x.$$

Hence,

$$\begin{aligned} u(t,x) &= \widetilde{u}(t,x) + \ell(t,x) \\ &= \sum_{k=1}^{\infty} \left[e^{-(k\pi)^2\alpha^2 t^2} A_k \sin(k\pi x) \right] + (e-1)x + 1. \end{aligned}$$

□

4.4 The Sturm-Liouville Eigenvalue Problem

Example 4.4.1

Solve the IBVP

$$\text{[PDE]} \quad \frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 1 \qquad \text{with } 0 < x < 1, t > 0$$

$$\text{[BCs]} \quad \begin{cases} \dfrac{\partial u}{\partial x}(t,0) = 0 \\ u(t,1) = t \end{cases} \qquad \text{with } t > 0$$

$$\text{[IC]} \quad u(0,x) = 0 \qquad \text{with } 0 < x < 1$$

Solution 1.

Introducing a lifting function:

$$u(t,x) = \widetilde{u}(t,x) + \ell(t,x).$$

Write $\ell(t,x) = P(t)x + Q(t)$, linear in x.

Then,

$$\frac{\partial \ell}{\partial x}(t,x) = P(t) = 0$$
$$\ell(t,x) = P(t) + Q(t) = t \implies Q(t) = t.$$

So, $\ell(t,x) = t$. Then, the problem becomes

$$\begin{aligned}\frac{\partial u}{\partial t} - \alpha^2\frac{\partial^2 u}{\partial x^2} &= \frac{\partial \widetilde{u}}{\partial t} + \frac{\partial \ell}{\partial t} - \alpha^2\left[\frac{\partial^2 \widetilde{u}}{\partial x^2} + \frac{\partial^2 \ell}{\partial x^2}\right] \\ &= \frac{\partial \widetilde{u}}{\partial t} + \cancel{1} - \alpha^2\frac{\partial^2 \widetilde{u}}{\partial x^2} = \cancel{1} \\ \frac{\partial \widetilde{u}}{\partial t} &- \alpha^2\frac{\partial^2 \widetilde{u}}{\partial x^2} = 0\end{aligned}$$

BCs:

$$\frac{\partial \widetilde{u}}{\partial x}(t,0) = 0, \widetilde{u}(t,1) = 0$$

IC:

$$\widetilde{u}(0,x) = 0$$

The solution is $\widetilde{u}(t,x) = 0$. So,

$$u(t,x) = \widetilde{u}(t,x) + \ell(t,x) = 0 + t = t.$$

Let's verify our solution. Solve using separation of variables.

$$\begin{cases} \dfrac{\partial u}{\partial t} - \alpha\dfrac{\partial^2 u}{\partial x^2} = 0 \\ \\ \dfrac{\partial u}{\partial x}(t,0) = 0, u(t,1) = 0 \\ u(0,x) = 0. \end{cases}$$

Assume $u(t,x) = T(t)X(x)$. Then, BCs are $X'(0) = 0$ and $X(1) = 0$. Since

$$X(x) = A\sin(\lambda x) + B\cos(\lambda x),$$

we have that

$$X'(x) = A\lambda\cos(\lambda x) - B\lambda\sin(\lambda x)$$

So, we have

$$\begin{aligned} X'(0) &= A\lambda \underbrace{\cos(0)}_{=1} - B\lambda \underbrace{\sin(0)}_{=0} \\ &= A\lambda = 0 \implies A = 0 \\ X(1) &= \underbrace{A}_{=0} \sin(\lambda) + B\cos(\lambda) = 0 \\ B\cos(\lambda) &= 0 \\ \cos(\lambda) &= 0 \\ \lambda_k &= \frac{(2k+1)\pi}{2}, \quad k = 0, 1, 2, \ldots \end{aligned}$$

So, $X_k(x) = B_k \cos(\lambda_k x)$. Then, the solution should look like

$$u(t,x) = \sum_{k=0}^{\infty} e^{-\lambda_k^2 \alpha^2 t} B_k \cos(\lambda_k x).$$

Imposing IC:

$$\begin{aligned} u(0,x) = \sum_{k=0}^{\infty} e^0 B_k \cos(\lambda_k x) &= 0 \\ B_k &= 0. \end{aligned}$$

Recall how we solved B_k using Fourier Series:

$$\begin{aligned} B_k &= 2\int_0^1 u_0(x)\cos(\lambda_k x)\,\mathrm{d}x, \quad \forall\, k = 0, 1, 2, \ldots \\ &= 0. \end{aligned}$$

□

Generally speaking, we apply separation of variable to solve the heat equation:

$$u(t,x) = T(t)X(x).$$

We have

$$\begin{aligned} X(x) &= A\sin(\lambda x) + B\cos(\lambda x) \\ X'(x) & A\lambda\cos(\lambda x) - B\lambda\sin(\lambda x) \end{aligned}$$

Note that,

$$\begin{aligned} X''(x) &= -A\lambda^2 \sin(\lambda x) - B\lambda^2 \cos(\lambda x) \\ &= -\lambda^2 \underbrace{[A \sin(\lambda x) + B\cos(\lambda x)]}_{X(x)} \\ &= -\lambda^2 X(x). \end{aligned}$$

So, we have

$$\boxed{X''(x) = -\lambda^2 X(x)}$$

This is an eigenvalue problem. [*Recall what we have with matrix and vectors: We want to find λ and $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Moreover, if $A = A^\top$ is a symmetric matrix, then eigenvalues of A are orthogonal to each other: $\langle \mathbf{x}_i, \mathbf{x}_j \rangle = 0 \quad$ for $i \neq j$.*] In this case, the second order derivative is a symmetric operator. [*Why? Think of integration by parts* $\int \frac{\mathrm{d}^2 X_j}{\mathrm{d}x^2} X_k \,\mathrm{d}x = -\int \frac{\mathrm{d}X_j}{\mathrm{d}x}\frac{\mathrm{d}X_k}{\mathrm{d}x}\,\mathrm{d}x$]. So, the *eigenfunctions* are orthogonal:

$$(X_i, X_j) = 0 \quad \text{if } i \neq j,$$

where $(X_i, X_j) = \int_0^1 X_i X_j \,\mathrm{d}x$ and λ^2 is called the *eigenvalue*.

Example 4.4.2 Orthogonal Functions

$X_k = A_k \sin(k\pi x)$ for $k = 0, 1, \ldots$ are orthogonal.

Proof 2.

$$(X_i, X_j) = \int_0^1 A_i \sin(i\pi x) A_j \sin(j\pi x)\,\mathrm{d}x = \begin{cases} 0 & \text{if } i \neq j \\ \frac{1}{2} & \text{if } i = j \end{cases}.$$

■

Example 4.4.3

Solve the IBVP

$$\begin{aligned} &\text{[PDE]} \quad \frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 && \text{with } 0 < x < 1, t > 0 \\ &\text{[BCs]} \quad \begin{cases} u(t,0) = 0 \\ \frac{\partial u}{\partial x}(t,1) = 0 \end{cases} && \text{with } t > 0 \\ &\text{[IC]} \quad u(0,x) = x && \text{with } 0 < x < 1 \end{aligned}$$

Solution 3.

Apply separation of variable: $u(t,x) = T(t)X(x)$. Then,

$$① \qquad T'(t) + \underbrace{\lambda^2}_{-k} \alpha^2 T(t) = 0$$

$$② \qquad \begin{cases} X''(x) + \lambda^2 X(x) = 0 \\ X(0) = 0, \; X'(1) = 0. \end{cases}$$

For ②, the solution looks like

$$\begin{aligned} X(x) &= A\sin(\lambda x) + B\cos(\lambda x) \\ X'(x) &= A\lambda\cos(\lambda x) - B\lambda\sin(\lambda x) \end{aligned}$$

Imposing the BCs:

$$\begin{aligned} X(0) &= A\lambda \underbrace{\sin(0)}_{=0} + B\underbrace{\cos(0)}_{=1} = 0 \implies B = 0 \\ X'(1) &= A\lambda\cos(\lambda) - \underbrace{B}_{=0}\sin(\lambda) = 0 \\ A\lambda\cos(\lambda) &= 0 \\ \cos(\lambda) &= 0 \qquad [A = 0 \textit{ is not interesting}] \\ \lambda_k &= \frac{(2k+1)\pi}{2}, \quad k = 0, 1, 2, \ldots \end{aligned}$$

So, the solution takes in the form

$$u(t,x) = \sum_{k=1}^{\infty} e^{-\lambda_k^2 \alpha^2 t} A_k \sin(\lambda_k x), \quad \text{where } \lambda_k = \frac{(2k+1)\pi}{2}.$$

Imposing IC:

$$\begin{aligned} u(0,x) &= \sum_{k=1}^{\infty} A_k \sin(\lambda_k x) = x \\ A_k &= 2\int_0^1 x\sin(\lambda_k x)\,\mathrm{d}x \\ \int x\sin(\lambda_k x) &= -\frac{1}{\lambda_k}\cos(\lambda_k x)x + \int \frac{1}{\lambda_k}\cos(\lambda x)\,\mathrm{d}x \\ &= -\frac{1}{\lambda_k}\cos(\lambda_k x)x + \frac{1}{\lambda_k^2}\sin(\lambda_k x) + C \end{aligned}$$

Therefore,

$$\begin{aligned} A_k = 2\int_0^1 x\sin(\lambda_k x)\,\mathrm{d}x &= 2\left[-\frac{1}{\lambda_k}\cos(\lambda_k x)x + \frac{1}{\lambda_k^2}\sin(\lambda_k x)\right]_0^1 \\ &= 2\left(-\frac{1}{\lambda_k}\cos(\lambda_k) + \frac{1}{\lambda_k^2}\sin(\lambda_k)\right) \\ &= -\frac{2}{\lambda_k}\underbrace{\cos(\lambda_k)}_{=0} + \frac{2}{\lambda_k^2}\sin(\lambda_k) \\ &= \frac{2}{\lambda_k^2}\sin\left(\frac{(2k+1)\pi}{2}\right) \\ &= \pm\frac{2}{\lambda_k^2}. \end{aligned}$$

□

Theorem 4.4.4 Sturm-Liouville (SL) Eigenvalue Problem
Let's consider the following problem

$$\begin{aligned} \text{[ODE]} \quad & (p(x)y')' - q(x)y + kr(x)y = 0 \\ \text{[BCs]} \quad & \alpha_0 y(0) + \beta_0 y'(0) = 0 \\ & \alpha_1 y(1) + \beta_1 y'(1) = 0. \end{aligned}$$

Let's assume:

- $p, p'q, r \in \mathcal{C}([0,1])$, and
- $p(x) > 0, r(x) > 0 \quad \forall\, x \in [0,1]$ (regularity).

Then,

1. All eigenvalues k_n are real,
2. If φ_i and φ_j are two eigenfunctions corresponding to $k_i \neq k_j$, then

$$(\varphi_i, \varphi_j)_r = \int_0^1 r\varphi_i\varphi_j\,\mathrm{d}w = 0, \qquad \text{(Orthogonal condition)}$$

   where r is a weight (constant)

3. To each eigenvalue corresponds ONLY one eigenfunction. Eigenfunctions are linearly independent, eigenvalues are real, and they form a ordered sequence

$$k_1 < k_2 < \cdots < k_N < \cdots .$$

In general, we write

$$u_0(x) = \sum_k^\infty \gamma_k X_k(x),$$

where $\gamma_k = \dfrac{(u_0, X_k)}{(X_k, X_k)}$. The 3^{rd} conclusion ensures the infinite sum in the Fourier series will converge.

Extension 4.1 (Not a Boundary Condition) *What if we don't have a boundary condition?*

$$\begin{aligned} &\textit{[PDE]} && \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 && \textit{with } 0 < x < 1, t > 0 \\ &\textit{[BCs]} && \begin{cases} u(t,0) = 0 \\ \dfrac{\partial u}{\partial x}\left(t, \dfrac{1}{2}\right) = 0 \end{cases} && \textit{with } t > 0 \\ &\textit{[IC]} && u(0,x) = x && \textit{with } 0 < x < 1 \end{aligned}$$

With our typical separation of variable, we can solve the problem for $0 < x < \frac{1}{2}$*, but how about the rest? We need to use* data assimilation *framework. That is, we want to find* $X_m(1)$ *that minimizes*

$$\left| \frac{\mathrm{d}X_{sol}}{\mathrm{d}x}\left(\frac{1}{2}\right) - \frac{\mathrm{d}X_m}{\mathrm{d}x}\left(\frac{1}{2}\right) \right|^2,$$

with $X_m(0) = 0$, $X_m(1)$*, and the PDE constraints.*

This is a constraint optimization, and we need Lagrangian multiplier to solve.

Example 4.4.5 SL Eigenvalue Problems

$$\begin{aligned} &\text{[PDE]} && \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 && \text{with } 0 < x < 1, t > 0 \\ &\text{[BCs]} && u(t,0) = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{(t,1)} + u(t,1) = 0 && \text{with } t > 0 \\ &\text{[IC]} && u(0,x) = u_0(x) && \text{with } 0 < x < 1 \end{aligned}$$

[*The physical interpretation of the second BC is that: the flux of heat is proportional to the temperature at the endpoint.*]

Solution 4.

Use separation of variables:

$$\begin{cases} T' - kT = 0 \\ X'' - kX = 0. \end{cases}$$

The BCs become

$$
\begin{aligned}
T(t)X(0) = 0 &\implies X(0) = 0 \\
T(t)X'(1) + T(t)X(1) = 0 &\implies X'(1) + X(1) = 0 \\
X(x) &= A\sin(\lambda X) + B\cos(\lambda x) \\
X(0) = A\underbrace{\sin(0)}_{=0} + B\underbrace{\cos(0)}_{=1} = 0 &\implies B = 0 \\
X'(1) + X(1) = A\lambda\cos(\lambda) - B\lambda\sin(\lambda) &+ A\sin(\lambda) + B\cos(\lambda) \\
= A\lambda\cos(\lambda) + A\sin(\lambda) &= 0 \\
\sin(\lambda) + \lambda\cos(\lambda) &= 0 \qquad [\textit{Since } A \neq 0] \\
\sin(\lambda) &= -\lambda\cos(\lambda) \\
\tan(\lambda) = \frac{\sin(\lambda)}{\cos(\lambda)} &= -\lambda
\end{aligned}
$$

Then, we can use numerical approaches to find λ's, and

$$
u_0(t,x) = \sum_{k=1}^{\infty} \gamma_k \varphi_k(x), \quad \text{where } \gamma_k = \frac{(u_0, \varphi_k)}{(\varphi_k, \varphi_k)} \text{ and } \varphi_k = \sin(\lambda_k x).
$$

Meanwhile, $T(t) = T_0 e^{-\lambda^2 \mu t}$. Then,

$$
u(t,x) = \sum_{k=1}^{\infty} \gamma_k e^{-\lambda_k^2 \mu t} \varphi_k(x) = \sum_{k=1}^{\infty} \frac{(u_0, \varphi_k)}{(\varphi_k, \varphi_k)} e^{-\lambda_k^2 \mu t} \sin(\lambda_k x).
$$

□

4.5 Nonhomogeneous Heat Equation

Consider the following IBVP:

$$
\begin{aligned}
&[\text{PDE}] \quad \frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = f(t,x) && \text{with } 0 < x < 1, t > 0 \\
&[\text{BCs}] \quad \begin{cases} a\dfrac{\partial u}{\partial x}(t,0) + \beta u(t,0) = 0 \\ \gamma\dfrac{\partial u}{\partial x}(t,1) + \delta u(t,1) = 0 \end{cases} && \text{with } t > 0 \\
&[\text{IC}] \quad u(0,x) = u_0(x) && \text{with } 0 < x < 1
\end{aligned}
$$

Assumption: $u(t,x) = \sum_k T_k(t)X_k(x)$. Compute the partial derivatives and plug-in:

$$\begin{aligned}\sum_k T_k'(t)X_k(x) - \alpha^2 \sum_k T_k(t)X_k''(x) &= f(t,x)\\ \sum_k \left[T_k'(t)X_k(x) - \alpha^2 T_k(t)X_k''(x)\right] &= f(t,x).\end{aligned}$$

Expresse $f(t,x) = \sum_k f_k(t)X_k(x)$, where $f_k(t) = \dfrac{(f(t,x), X_k(x))}{(X_k(x), X_k(x))}$. The equation becomes

$$\sum_k \left[T_k'(t)X_k(x) - \alpha^2 T_k(t)X_k''(x)\right] = \sum_k f_k(t)X_k(x).$$

From Separation of Variable on the homogeneous equation and the SL Eigenvalue Problem,

$$X_k''(x) = -\lambda^2 X_k(x).$$

Substitute:

$$\begin{aligned}\sum_k \left[T_k'(t)X_k(x) + \alpha^2\lambda_k^2 T_k(t)X_k(x)\right] &= \sum_k f_k(t)X_k(x)\\ \sum_k \underbrace{\left[T_k'(t) + \alpha^2\lambda_k^2 T_k(t)\right]} X_k(x) &= \sum_k \underbrace{f_k(t)} X_k(x)\end{aligned}$$

Comparing terms, we get a nonhomogeneous first-order ODE to solve:

$$T_k'(t) + \alpha^2\lambda_k^2 T_k(t) = f_k(t)$$

Apply integrating factors, we have

$$\mu(t) = e^{\int \alpha^2\lambda_k^2 \, \mathrm{d}t} = e^{\alpha^2\lambda_k^2 t}.$$

So, the general solution is

$$\begin{aligned}T_k(t) &= e^{-\alpha^2\lambda_k^2 t}\left[\int_0^t e^{\alpha^2\lambda_k^2 s} f_k(s)\,\mathrm{d}s + T_k(0)\right]\\ &= \int_0^t e^{-\alpha^2\lambda_k^2 t} e^{\alpha^2\lambda_k^2 s} f_k(s)\,\mathrm{d}s + e^{-\alpha^2\lambda_k^2 t}T_k(0)\\ &= \int_0^t e^{-\alpha^2\lambda_k^2 (t-s)} f_k(s)\,\mathrm{d}s + e^{-\alpha^2\lambda_k^2 t}T_k(0)\end{aligned}$$

Therefore,

$$\begin{aligned} u(t,x) &= \sum_k T_k(t)X_k(x) \\ &= \sum_k \Bigg[\underbrace{\int_0^t e^{-\alpha^2\lambda_k^2(t-s)} f_k(s)\,\mathrm{d}s}_{\substack{\text{additional term for} \\ \text{the nonhomogeneous part}}} + \underbrace{e^{-\alpha^2\lambda_k^2 t}T_k(0)}_{\substack{\text{solution to the} \\ \text{homogeneous} \\ \text{part}}} \Bigg] X_k(x) \end{aligned}$$

Finally, impose the initial condition:

$$\begin{aligned} u(0,x) &= \sum_k \Bigg[\underbrace{\int_0^0 e^{-\alpha^2\lambda_k^2(0-s) f_k(s)\,\mathrm{d}s}}_{=0} + \underbrace{e^{-\alpha^2\lambda_k^2 0}T_k(0)}_{=T_k(0)} \Bigg] X_k(x) \\ &= \sum_k T_k(0)X_k(x) = u_0(x). \end{aligned}$$

So,

$$T_k(0) = \frac{(u_0(x), X_k(x))}{(X_k(x), X_k(x))}.$$

Example 4.5.1

$$\begin{aligned} &\text{[PDE]} && \frac{\partial u}{\partial t} - 2\frac{\partial^2 u}{\partial x^2} = f(t,x) && \text{with } 0 < x < 1, t > 0 \\ &\text{[BCs]} && \begin{cases} u(t,0) = 0 \\ u(t,1) = 0 \end{cases} && \text{with } t > 0 \\ &\text{[IC]} && u(0,x) = u_0(x) && \text{with } 0 < x < 1 \end{aligned}$$

Solution 1.

Assume that $u(t,x) = \sum_k T_k(x)X_k(x)$. We need to solve the SL Eigenvalue Problem

$$\begin{cases} X''(x) + \lambda^2 X(x) = 0 \\ X(0) = 0,\ X(1) = 0. \end{cases}$$

Assume $X(x) = A\sin(\lambda x) + B\cos(\lambda x)$. Then,

$$X(0) = A\sin(0) + B\cos(0) = 0 \implies B = 0$$

$$X(1) = A\sin(\lambda) + B\cos(\lambda) = 0 \implies A\sin(\lambda) = 0.$$

Since $A \neq 0$, it must be $\sin(\lambda) = 0 \implies \lambda_k = k\pi, \quad k = 1, 2, \ldots$. Then,

$$X_k(x) = \sin(\lambda_k x) = \sin(k\pi x) \quad \text{and} \quad \lambda_k^2 = (k\pi)^2,\ X_k''(x) = -(k\pi)^2 \sin(k\pi x).$$

So, the equation becomes

$$\sum_k \left[T_k'(t) + 2(k\pi)^2 T_k(t)\right] X_k(x) = \sum_k f_k(t) X_k(x),$$

where

$$\begin{aligned} f_k(t) = \frac{(f(t,x), X_k(x))}{(X_k(x), X_k(x))} = 2\int_0^1 t\sin(k\pi x)\,\mathrm{d}x &= 2t\int_0^1 \sin(k\pi x)\,\mathrm{d}x \\ &= 2t\left[-\frac{1}{k\pi}\cos(k\pi x)\right]_0^1 \\ &= \frac{2t}{k\pi}[1-\cos(k\pi)]. \end{aligned}$$

We then have the following ODE to solve:

$$\begin{aligned} T_k'(t) + 2(k\pi)^2 T_k(t) &= \frac{2t}{k\pi}[1-\cos(k\pi)] \\ T_k(0) = A_k &= 2\int_0^1 u_0(x)\sin(k\pi x)\,\mathrm{d}x. \end{aligned}$$

Integrating factor: $\mu(t) = e^{\int 2(k\pi)^2\,\mathrm{d}t} = e^{2(k\pi)^2 t}$. Then,

$$T_k(t) = e^{-2(k\pi)^2 t}\left[\frac{2}{k\pi}[1-\cos(k\pi)]\int_0^t s e^{2(k\pi)^2 s}\,\mathrm{d}s + T_k(0)\right]$$

$$\begin{aligned} \int_0^t s e^{2(k\pi)^2 s}\,\mathrm{d}s &= \left[\frac{1}{2(k\pi)^2} s e^{2(k\pi)^2 s}\right]_0^t - \int_0^t \frac{1}{2(k\pi)^2} e^{2(k\pi)^2 s}\,\mathrm{d}s \\ &= \frac{1}{2(k\pi)^2} t e^{2(k\pi)^2 t} - \left[\frac{1}{4(k\pi)^4} e^{2(k\pi)^2 s}\right]_0^t \\ &= \frac{1}{2(k\pi)^2} t e^{2(k\pi)^2 t} - \frac{1}{4(k\pi)^4}\left[e^{2(k\pi)^2 t} - 1\right] \end{aligned}$$

$$T_k(t) = \frac{2}{k\pi}[1-\cos(k\pi)]\left[\frac{t}{2(k\pi)^2} - \frac{1}{4(k\pi)^4} + \frac{1}{4(k\pi)^4} e^{-2(k\pi)^2 t}\right] + e^{-2(k\pi)^2 t} T_k(0),$$

where $T_k(0) = A_k = 2\int_0^1 u_0(x)\sin(k\pi x)\,\mathrm{d}x$. □

5 Bounded Wave Equation

Remark. We can generalize the method of Separation of Variable. To solve

$$P_t u = \Delta_x u,$$

where P_t is some first-order differential operator with respect to time and Δ_x is the Laplacian with respect to space.
Let's assume $u(t,x) = T(t)X(x)$. Then,

$$\begin{aligned} P_t[u(t,x)] &= P_t[T(t)X(x)] = P_t[T(t)]X(x) \\ \Delta_x u(t,x) &= \Delta_x[T(t)X(x)] = T(t)\Delta_x[X(x)]. \end{aligned}$$

So, the PDE becomes

$$\begin{aligned} P_t[T(t)]X(x) &= T(t)\Delta_x[X(x)] \\ \frac{P_t[T(t)]}{T(t)} &= \frac{\Delta_x[X(x)]}{X(x)} = k \end{aligned}$$

Then, $P_t[T(t)] = kT(t)$ and $\Delta_x[X(x)] = kX(x)$.

5.1 No Damping Force

Consider the IBVP

$$\begin{aligned} &\text{[PDE]} && \frac{\partial^2 u}{\partial t^2} - \gamma^2 \frac{\partial^2 u}{\partial x^2} = 0 && \text{with } 0 < x < 1, t > 0 \\ &\text{[BCs]} && \begin{cases} u(t,0) = 0 \\ u(t,1) = 0 \end{cases} && \text{with } t > 0 \\ &\text{[IC]} && \begin{cases} u(0,x) = u_0(x) \\ \dfrac{\partial u}{\partial t}(0,x) = v_0(x) \end{cases} && \text{with } 0 < x < 1 \end{aligned}$$

Use Separation of Variable. Assume $u(t,x) = T(t)X(x)$. Then,

$$T''(t)X(x) - \gamma^2 T(t)X''(x) = 0.$$

Divide by $\gamma^2 u(t,x) = \gamma^2 T(t)X(x)$:

$$\frac{T''(t)\cancel{X(x)}}{\gamma^2 T(t)\cancel{X(x)}} - \frac{\cancel{\gamma^2 T(t)}X''(x)}{\cancel{\gamma^2 T(t)}X(x)} = 0$$
$$\frac{T''(t)}{\gamma^2 T(t)} - \frac{X''(x)}{X(x)} = 0 \implies \frac{T''(t)}{\gamma^2 T(t)} = \frac{X''(x)}{X(x)} = k = -\lambda^2 < 0.$$

So, we have two SL Eigenvalue Problems:

$$① \begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0, \ X(1) = 0 \end{cases} \qquad ② \begin{cases} T''(t) + \lambda^2\gamma^2 T(t) = 0 \\ T(0) = u_0(x) \\ T'(0) = v_0(x) \end{cases}$$

From ①:

$$X_k = \sin(k\pi x), \quad \lambda_k = k\pi, \quad \text{and } \lambda_k^2 = (k\pi)^2.$$

From ②:

$$\begin{aligned} T_k(t) &= A_k \sin(\lambda\gamma t) + B_k \cos(\lambda\gamma t) \\ &= A_k \sin(k\pi\gamma t) + B_k \cos(k\pi\gamma t). \end{aligned}$$

Then,

$$u_k(t,x) = T_k(t)X_k(x) = \sin(k\pi x)[A_k \sin(k\pi\gamma t) + B_k \cos(k\pi\gamma t)].$$

[*For a general SL-Eigenvalue problem,*

$$X_k(x) = C_k \sin(\lambda_k x) + D_k \cos(\lambda_k x).$$

So,

$$\begin{aligned} u_k(t,x) &= T_k(t)X_k(X) \\ &= [A_k \sin(\lambda_k\gamma t) + B_k \cos(\lambda_k\gamma t)][C_k \sin(\lambda_k x) + D_k \cos(\lambda_k x)] \end{aligned}$$

] Now, to apply ICs:

$$T_k'(t) = \gamma\lambda_k A_k \cos(\gamma\lambda_k t) - \gamma\lambda_k B_k \sin(\gamma\lambda_k t).$$

So, $T_k(0) = B_k$ and $T_k'(0) = \gamma\lambda_k A_k$. Then,

$$\begin{aligned}
u(0,x) = \sum_k T_k(0)X_k(x) &= u_0(x) \\
\sum_k B_k \sin(k\pi x) &= u_0(x) \\
\implies B_k &= \frac{(u_0(x), X_k(x))}{(X_k(x), X_k(x))} = 2\int_0^1 u_0(x)\sin(k\pi x)\,\mathrm{d}x. \\
\frac{\partial}{\partial t}u(0,x) = \sum_k T_k(0)X_k(x) &= v_0(x) \\
\sum_k \gamma\lambda_k A_k \sin(k\pi x) &= v_0(x)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\gamma\lambda_k A_k &= \frac{(v_0(x), X_k(x))}{(X_k(x), X_k(x))} = 2\int_0^1 v_0(x)\sin(k\pi x)\,\mathrm{d}x \\
A_k &= \frac{2}{\gamma\lambda_k}\int_0^1 v_0(x)\sin(k\pi x)\,\mathrm{d}x \\
&= \frac{2}{k\pi\gamma}\int_0^1 v_0(x)\sin(k\pi x)\,\mathrm{d}x.
\end{aligned}$$

Hence, the solution is

$$u(t,x) = \sum_{k=1}^{\infty} \sin(k\pi x)[A_k \sin(k\pi\gamma t) + B_k\cos(k\pi\gamma t)],$$

where

$$A_k = \frac{2}{k\pi\gamma}\int_0^1 v_0(x)\sin(k\pi x)\,\mathrm{d}x \quad \text{and} \quad B_k = 2\int_0^1 u_0(x)\sin(k\pi x)\,\mathrm{d}x.$$

Definition 5.1.1 (k-th Mode of Vibration). The k-th term of the solution is called the *k-th mode of vibration k-th harmonie.*

$$\begin{aligned}
u_k(t,x) &= \sin(k\pi x)[A_k\sin(k\pi\gamma t) + B_k\cos(k\pi\gamma t)] \\
&= R_k \sin(k\pi x)\cos[k\pi\gamma(t-\delta_k)] && [\textit{trig. identity}]
\end{aligned}$$

We call R_k the *amplitude*, and δ_k the *phase angle*. Meanwhile, the *frequency* of k-th mode is defined as

$$\omega_k = k\pi\gamma = k\pi\sqrt{\frac{T}{\rho}}.$$

Remark. If we have non-homogeneous boundary conditions, we use lifting functions.

5.2 With Damping

Consider the IBVP

$$\text{[PDE]} \quad \frac{\partial^2 u}{\partial t^2} - \gamma^2 \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial u}{\partial t} = 0 \qquad \text{with } 0 < x < 1, t > 0$$

$$\text{[BCs]} \quad \begin{cases} u(t,0) = 0 \\ u(t,1) = 0 \end{cases} \qquad \text{with } t > 0$$

$$\text{[IC]} \quad \begin{cases} u(0,x) = u_0(x) \\ \dfrac{\partial u}{\partial t}(0,x) = v_0(x) \end{cases} \qquad \text{with } 0 < x < 1$$

Assume $u(t,x) = T(t)X(x)$. Then,

$$\begin{aligned} T''(t)X(x) - \gamma^2 T(t)X''(x) + \beta T'(t)X(x) &= 0 \\ \frac{T''(t)\cancel{X(x)}}{\gamma^2 T(t)\cancel{X(x)}} - \frac{\cancel{\gamma^2 T(t)}X''(x)}{\cancel{\gamma^2 T(t)}X(x)} + \beta \frac{T'(t)\cancel{X(x)}}{\gamma^2 T(t)\cancel{X(x)}} &= 0 \\ \frac{T''(t) + \beta T'(t)}{\gamma^2 T(t)} = \frac{X''(x)}{X(x)} &= k = -\lambda^2. \end{aligned}$$

So, we have

$$① \; T''(t) + \beta T'(t) + \lambda^2 \gamma^2 T(t) = 0 \quad ② \begin{cases} X''(x) + \lambda^2 X(x) = 0 \\ X(0) = 0, \; X(1) = 0. \end{cases}$$

② is an SL Eigenvalue Problem, and we have $X_k(X) = \sin(k\pi x)$. Now, solve ①, using characteristic polynomial:

$$p(r) = r^2 + \beta r + \lambda^2\gamma^2 = 0 \implies r_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4\lambda^2\gamma^2}}{2}.$$

We want to have complex conjugate roots, so

$$\beta^2 - 4\lambda^2\gamma^2 < 0 \implies \beta \ll \gamma.$$

[*If β is large, we get too much damping. In such a case, no oscillations anymore.*]

Denote

$$\frac{\sqrt{\beta^2 - 4\lambda^2\gamma^2}}{2} = \mathrm{i}\delta_k.$$

Then,

$$T_k(t) = e^{(-\beta/2)t}[A_k \sin(\delta_k t) + B_k \cos(\delta_k t)].$$

Finally,

$$\begin{aligned} u(t,x) &= \sum_{k=1}^{\infty} T_k(t) X_k(x) \\ &= \sum_{k=1}^{\infty} e^{(-\beta/2)t}[A_k \sin(\delta_k t) + B_k \cos(\delta_k t)] \sin(k\pi). \end{aligned}$$

5.3 With External Force

Consider the IBVP

$$\begin{aligned}
&\text{[PDE]} \quad \frac{\partial^2 u}{\partial t^2} - \gamma^2 \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial u}{\partial t} = f(t,x) && \text{with } 0 < x < 1, t > 0 \\
&\text{[BCs]} \quad \begin{cases} u(t,0) = 0 \\ u(t,1) = 0 \end{cases} && \text{with } t > 0 \\
&\text{[IC]} \quad \begin{cases} u(0,x) = u_0(x) \\ \dfrac{\partial u}{\partial t}(0,x) = v_0(x) \end{cases} && \text{with } 0 < x < 1
\end{aligned}$$

Assume $u(t,x) = T(t)X(x)$. Then,

$$T''(t)X(x) - \gamma^2 T(t) X''(x) + \beta T'(t) X(x) = f(t,x).$$

From the Sl-Eigenvalue PRoblem, $X''(x) = -\lambda X(x)$. So,

$$\begin{aligned} T''(t)X(x) + \gamma^2 \lambda^2 T(t) X(x) + \beta T'(t) X(x) &= \widetilde{f}(t) X(x) && \text{[\textit{project} } f(t,x) \textit{ on } X(x)] \\ \big[T''(t) + \beta T'(t) + \gamma^2 \lambda^2 T(t)\big] X(x) &= \widetilde{f}(t) X(x), \end{aligned}$$

where

$$\widetilde{f}_k(t) = \frac{(f(t,x), X_k(x))}{(X_k(x), X_k(x))} = 2 \int_0^1 f(t,x) \sin(k\pi x)\, dx.$$

So, for every k, we want to solve:

$$T_k''(t) + \beta T_k'(t) + \gamma^2 \lambda^2 T_k(t) = f_k(t).$$

This is a second-order ODE that requires a particular solution:

$$T_k(t) = e^{(-\beta/2)t}[A_k \sin(\delta_k t) + B_k \cos(\delta_k t)] + \widetilde{T}_k(t),$$

where $\widetilde{T}_k(t)$ is a particular solution for the non-homogeneous problem:

- Method of Undetermined Coefficient
- Reduction of Orders
- Variational of Parameters:

$$\widetilde{T}_k(t) = -T_1(t)\int \frac{T_2(t)f_k(t)}{\mathbf{W}[T_1(t),T_2(t)]}\,\mathrm{d}t + T_2(t)\int \frac{T_1(t)f_k(t)}{\mathbf{W}[T_1(t),T_2(t)]}\,\mathrm{d}t$$

5.4 Boundary Conditions

Generic BCs look like the following:

$$\begin{cases} \alpha\dfrac{\partial u}{\partial x}(t,0) + \beta u(t,0) = 0 \\ \\ \gamma\dfrac{\partial u}{\partial x}(t,1) + \delta u(t,1) = 0 \end{cases}$$

We could have three possible types of BCs:

- Dirichlet BCs:
$$\begin{cases} u(t,0) = f(t) \\ u(t,1) = g(t) \end{cases}$$
Here we have fixed points as our endpoints.

- Neumann BCs:
$$\begin{cases} \dfrac{\partial u}{\partial x}(t,0) \\ \\ \dfrac{\partial u}{\partial x}(t,1) \end{cases}$$
Note that we don't have fixed points here. We only fix slopes.

- Robin BCs:
$$\begin{cases} \dfrac{\partial u}{\partial x}(t,0) + \alpha_1 u(t,0) \\ \\ \dfrac{\partial u}{\partial x}(t,1) + \alpha_2 u(t,1) \end{cases}$$
Here we have a combination of Dirichlet and Neumann BCs.

Example 5.4.1

Solve the following IBVP:

$$\text{[PDE]} \quad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \alpha x \qquad \text{with } 0 < x < 1, t > 0$$

$$\text{[BCs]} \quad \begin{cases} u(t,0) = 0 \\ u(t,1) = 0 \end{cases} \qquad \text{with } t > 0$$

$$\text{[IC]} \quad \begin{cases} u(0,x) = u_0(x) \\ \dfrac{\partial u}{\partial t}(0,x) = 0 \end{cases} \qquad \text{with } 0 < x < 1$$

Solution 1.

Case I $\alpha = 0$:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0.$$

This is a homogeneous wave equation with Dirichlet BCs. By separation of variables, $u(t,x) = T(t)X(x)$. Then, we have SL Eigenvalue problems:

$$① \begin{cases} X''(x) + \lambda^2 X(x) = 0 \\ X(0) = 0,\ X(1) = 0 \end{cases} \qquad ② \begin{cases} T''(t) + \lambda^2 T(t) = 0 \\ T(0) = \widetilde{u}_0,\ T'(0) = \widetilde{v}_0 = 0. \end{cases} \qquad [\gamma = 1 \implies \lambda\gamma = \lambda]$$

Solving ①: $X_k(x) = \sin(k\pi x)$. So, $\lambda_k = k\pi \implies \lambda_k^2 = (k\pi)^2$ for $k = 1, 2, \ldots$.

Solving ②:

$$T_k(t) = A_k \sin(\lambda_k t) + B_k \cos(\lambda_k t)$$
$$T_k'(t) = \lambda_k A_k \cos(\lambda_k t) - \lambda_k B_k \sin(\lambda_k t).$$

Imposing IC, we get

$$T_k(0) = B_k \cos(0) = B_k$$
$$T_k'(0) = \lambda_k A_k \cos(0) = \lambda_k A_k = k\pi A_k$$

Since $u(t,x) = \sum_{k=1}^{\infty} T_k(t)X_k(x)$, we have [$k$ *starts from* 1 *b/c* $X_0(x) = 0$. *We don't see anything.*]

$$u(0,x) = \sum_{k=1}^{\infty} T_k(0)X_k(x) = \sum_{k=1}^{\infty} B_k \sin(k\pi x) = u_0(x),$$

where

$$B_k = \frac{(u_0(x), X_k(x))}{(X_k(x), X_k(x))} = 2\int_0^1 u_0(x)\sin(k\pi x)\,\mathrm{d}x.$$

Meanwhile,

$$\frac{\partial u}{\partial t}(0,x) = \sum_{k=1}^{\infty} T_k'(0)X_k(x) = \sum_{k=1}^{\infty} k\pi A_k \sin(k\pi x) = v_0(x) = 0,$$

where

$$k\pi A_k = \frac{(v_0(x), X_k(x))}{(X_k(x), X_k(x))} = 2\int_0^1 0\cdot\sin(k\pi x)\,\mathrm{d}x = 0 \implies A_k = 0.$$

So, $T(t) = A_k\sin(\lambda_k t) + B_k\cos(\lambda_k t) = B_k\cos(k\pi t)$ for $k = 1, 2, \ldots$.

Hence, the solution

$$u(t,x) = \sum_{k=1}^{\infty} B_k\cos(k\pi t)\sin(k\pi x),$$

where

$$B_k = 2\int_0^1 u_0(x)\sin(k\pi t)\,\mathrm{d}x.$$

Case II $\alpha \neq 0$:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \alpha x.$$

From separation of variable, assume $u(t,x) = T(t)X(x)$. Then, we get the same SL Eigenvalue Problem:

$$\begin{cases} X''(x) + \lambda^2 X(x) = 0 \\ X(0) = 0,\ X(1) = 0 \end{cases} \implies \lambda_k = k\pi, \lambda_k^2 = (k\pi)^2,\ X_k = \sin(k\pi x).$$

The PDE becomes

$$\begin{aligned} T''(t)X(x) - T(t)X''(x) &= \alpha x \\ T''(t)X(x) - T(t)(-\lambda^2 X(x)) &= \alpha x && [X''(x) + \lambda^2 X(x) = 0] \\ T''(t)X(x) + \lambda^2 T(t)X(x) &= \alpha x \\ \left[T''(t) + \lambda^2 T(t)\right]X(x) &= \alpha x \\ \sum_{k=1}^{\infty}\left[T_k''(t) + \lambda_k^2 T_k(t)\right]X_k(x) &= \alpha x = \sum_{k=1}^{\infty} C_k X_k(x), \end{aligned}$$

where C_k does not depend on t because αx has no t terms. So,

$$C_k = \frac{(\alpha x, X_k(x))}{(X_k(x), X_k(x))} = 2\int_0^1 \alpha x \sin(k\pi x)\,\mathrm{d}x.$$

Compare the terms, we get

$$\begin{aligned} T_k''(t) + \lambda_k^2 T_k(t) &= C_k \\ T_k''(t) + (k\pi)^2 T_k(t) &= C_k && [\lambda_k^2 = (k\pi)^2] \end{aligned}$$

So,

$$T_k(t) = \underbrace{A_k \sin(k\pi t) + B_k \cos(k\pi t)}_{\substack{\text{general solution for} \\ \text{homogeneous part}}} + \underbrace{T_k^p(t)}_{\substack{\text{particular solution} \\ \text{related to RHS}}}$$

In this example, we would guess $T_k^p(t)$ is a constant. By method of undetermined coefficients, we get

$$T_k^p(t) = \frac{C_k}{(k\pi)^2}.$$

Imposing ICs:

$$\begin{aligned} T_k(0) &= A_k \sin(0) + B_k \cos(0) + \frac{C_k}{(k\pi)^2} = B_k + \frac{C_k}{(k\pi)^2} \\ T_k'(t) &= (k\pi) A_k \cos(k\pi t) - (k\pi) B_k \sin(k\pi t) \\ T_k'(0) &= (k\pi) A_k \cos(0) - (k\pi) B_k \sin(0) = 0. \end{aligned}$$

Hence,

$$(k\pi) A_k = 0 \implies A_k = 0.$$

Since $u(t,x) = \sum_{k=1}^{\infty} T_k(t) X_k(x)$, we have

$$\begin{aligned} u(0,x) &= \sum_{k=1}^{\infty} T_k(0) X_k(x) = u_0 \\ \sum_{k=1}^{\infty} \left(B_k + \frac{C_k}{(k\pi)^2} \right) X_k(x) &= u_0(x) = \sum_{k=1}^{\infty} D_k X_k(x), \end{aligned}$$

where

$$D_k = \frac{(u_0(x), X_k(x))}{(X_k(x), X_k(x))} = 2\int_0^1 u_0(x) \sin(k\pi x)\,\mathrm{d}x.$$

So,

$$B_k + \frac{C_k}{(k\pi)^2} = D_k \implies B_k = D_k - \frac{C_k}{(k\pi)^2}.$$

Hence,

$$T_k(t) = B_k \cos(k\pi t) + \frac{C_k}{(k\pi)^2}.$$

The final solution is

$$\begin{aligned} u(t,x) &= \sum_{k=1}^{\infty} T_k(t) X_k(x) \\ &= \sum_{k=1}^{\infty} \left(B_k \cos(k\pi t) + \frac{C_k}{(k\pi)^2} \right) \sin(k\pi x), \end{aligned}$$

where

$$\begin{aligned} B_k &= D_k - \frac{C_k}{(k\pi)^2} \\ C_k &= 2\alpha \int_0^1 x \sin(k\pi x)\, \mathrm{d}x, \quad \text{and} \\ D_k &= 2 \int_0^1 u_0(x) \sin(k\pi x)\, \mathrm{d}x. \end{aligned}$$

□

6 Laplace Equation on Circular Domains

6.1 Polar Coordinates

Recall from Heat Equation: $P_t u = \Delta u$. Now, we want to eliminate the derivatives of t. So, we just have $\Delta u = f$. But if u is singular variable,

$$\Delta u = \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f,$$

and we get just a second order ODE. It is not interesting. So, suppose $u = u(x, y)$. Then,

$$\boldsymbol{\nabla} u = \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix}.$$

Recall the divergence operator:

$$\boldsymbol{\nabla} \cdot \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}.$$

So,

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} u = \boldsymbol{\nabla} \cdot \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{bmatrix} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Delta x,$$

where Δ is called the Laplacian operator. Then , we have

$$\begin{aligned} -\Delta u = f \qquad &\leftarrow \text{Poisson Equation} \\ \Delta u = 0 \qquad &\leftarrow \text{Laplace Equation} \end{aligned}$$

Usually, Laplace equations are defined cover circular domains. So, it is natural for us to use the polar coordinates.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}.$$

Then, our goal is to write $u(r(x, y), \theta(x, y))$, where

$$\begin{cases} r = \sqrt{x^2 + y^2} & 0 \le r \le R \\ \theta = \arctan\left(\frac{y}{x}\right) & 0 \le \theta \le 2\pi. \end{cases}$$

Since Laplacian in Cartesian is given by

$$\Delta u(x, y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

we can apply chain rule and get

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right)\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial x}\right) \\
&= \left[\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\right)\frac{\partial \theta}{\partial x}\right]\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} \\
&\qquad + \left[\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right)\frac{\partial \theta}{\partial x}\right]\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \theta} + \frac{\partial^2 \theta}{\partial x^2} \\
&= \left[\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial \theta \partial r}\frac{\partial \theta}{\partial x}\right]\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} \\
&\qquad + \left[\frac{\partial^2 u}{\partial r \partial \theta}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial \theta^2}\frac{\partial \theta}{\partial x}\right]\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2} \\
&= \frac{\partial^2 u}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial \theta \partial r}\frac{\partial \theta}{\partial x}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2 + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2}
\end{aligned}
$$

Finding derivatives, we have

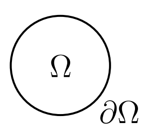
$$
\frac{\partial r}{\partial x} = \cos\theta, \quad \frac{\partial^2 r}{\partial x^2} = \frac{\sin^2\theta}{r}, \quad \frac{\partial \theta}{\partial x} = \frac{1}{r}\sin\theta, \quad \frac{\partial^2 \theta}{\partial x^2} = -\frac{2\cos\theta\sin\theta}{r^2}.
$$

So, we can find

$$
\Delta u(r,\theta) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.
$$

Now, let's define the Laplace Equation problem again:

- Ω: Disc of radius 1, centered at $(0,0)$
- Γ: $\partial\Omega$, the boundary of Ω.



Then, the Laplace Equation with boundary conditions is given by

$$
\begin{cases}
\text{[PDE]} & \Delta u = 0 \quad \text{on } \Omega \\
\text{[BC]} & u = g \quad \text{on } \Gamma = \partial\Omega
\end{cases}
$$

Transforming to polar coordinate, we have

$$
\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.
$$

To apply separation of variable, assume $u(r,\theta) = R(r)\Theta(\theta)$. Then,

$$\Delta u = R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(R)\Theta''(\theta) = 0.$$

Multiply by r^2 and divide by $R(r)\Theta(\theta)$:

$$\begin{aligned} r^2\frac{R''(r)\Theta(\theta)}{R(r)\Theta(\theta)} + \frac{r^2R'(r)\Theta(\theta)}{rR(r)\Theta(\theta)} + \frac{r^2R(r)\Theta''(\theta)}{r^2R(r)\Theta(\theta)} &= 0 \\ \frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} &= 0 \\ \frac{r^2R''(r) + rR'(r)}{R(r)} &= -\frac{\Theta''(\theta)}{\Theta(\theta)} = k = \lambda^2 \end{aligned}$$

So, we have two ODEs to solve:

$$① \quad r^2R''(r) + rR'(r) - \lambda^2R(r) = 0, \qquad ② \quad \Theta''(\theta) + \lambda^2\Theta(\theta) = 0.$$

Solving ①: guess $R_n(r) = r^n$. Then, $R'(r) = nr^{n-1}$ and $R''(r) = n(n-1)r^{n-2}$. So,

$$\begin{aligned} r^2n(n-1)r^{n-2} + rnr^{n-1} - \lambda^2r^n &= 0 \\ (n^2-n)r^n + nr^n - \lambda^2r^n &= 0 \\ (n^2-n+n-\lambda^2)r^n &= 0 \\ (n^2-\lambda^2)r^n &= 0 \\ \implies n^2-\lambda^2 = 0 \implies n^2 = \lambda^2 \implies &\boxed{\lambda_{1,2} = \pm n}. \end{aligned}$$

[*We are solving for the eigenvalues here. Eventually, we will have a sum of* $R_n(r)\Theta_n(\theta)$.] Here,

$$\text{FSS} = \left\{r^n, r^{-n}\right\}.$$

However, we need to also consider the special case: what if $\lambda^2 = 0$? Then,

$$r^2R''(R) + rR'(r) = 0.$$

Denote $Z(r) = R'(r)$ and $Z'(r) = R''(r)$, [*Reduction of Order*] we have

$$\begin{aligned} r^2Z'(r) + rZ(r) &= 0 \\ Z'(r) + \frac{1}{r}Z(r) &= 0 \\ \implies R'(r) = Z(r) &= c_1e^{-\int \frac{1}{r}\,\mathrm{d}r} = c_1e^{-\ln r} = \frac{c_1}{r}. \end{aligned}$$

Hence,

$$R(r) = \int R'(r)\,\mathrm{d}r = \int \frac{c_1}{r}\,\mathrm{d}r = c_1 \ln r + c_2$$

So, FSS $= \{\ln r, 1\}$.

In this case, since $0 < r < 1$, we don't want singularities at $r = 0$. So, we will eliminate non-singular cases. That is, r^{-n} and $\ln r$ are omitted. In summary,

$$R(r) = \begin{cases} Cr^n & \text{if } \lambda = n \neq 0 \\ 1 & \text{if } \lambda = n = 0. \end{cases}$$

Solving ②, we solve an SL Eigenvalue Problem:

$$\begin{aligned} \Theta''(\theta) + \lambda^2\Theta(\theta) = 0 \\ \Theta(\theta) &= A\cos(\lambda\theta) + B\sin(\lambda\theta) \\ &= A\cos(n\theta) + B\sin(n\theta) && [\lambda = n \textit{ from } ①] \end{aligned}$$

Hence, the n-th general solution on Ω is given by

$$\begin{aligned} u_n(r,\theta) &= R_n(r)\Theta_n(\theta) \\ &= Cr^n[A_n\cos(n\theta) + B_n\sin(n\theta)] \\ &= r^n[A_n\cos(n\theta) + B_n\sin(n\theta)] && [\textit{Drop constant } C] \end{aligned}$$

So, the general solution is

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n[A_n\cos(n\theta) + B_n\sin(n\theta)].$$

At the boundary,

$$u(1,\theta) = g(\cos\theta, \sin\theta).$$

[$g(x,y) = g(r\cos\theta, r\sin\theta)$, *but here* $r = 1$.] Let's write $g(\cos\theta,\sin\theta) = \widehat{g}(\theta)$ on Γ. Then, project $\widehat{g}(\theta)$ onto the same functional space:

$$A_0 = \frac{(\widehat{g}(\theta), \cos(0\cdot n))}{(\cos(0\cdot n), \cos(0\cdot n))} = \frac{\displaystyle\int_0^{2\pi} \cos(0\cdot\theta)\widehat{g}(\theta)\,\mathrm{d}\theta}{\displaystyle\int_0^{2\pi} \cos^2(0\cdot\theta)\,\mathrm{d}\theta} = \frac{1}{2\pi}\int_0^{2\pi} \widehat{g}(\theta)\,\mathrm{d}\theta.$$

[*Note that* $u(0,0) = A_0 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\widehat{g}(\theta)\,\mathrm{d}\theta$. *So, the energy at the center is the average energy on* *the boundary.*] We also can find

$$A_n = \frac{(\widehat{g}(\theta), \cos(n\theta))}{(\cos(n\theta), \cos(n\theta))} = \frac{\displaystyle\int_0^{2\pi} \cos(n\theta)\widehat{g}(\theta)\,\mathrm{d}t}{\displaystyle\int_0^{2\pi} \cos^2(n\theta)\,\mathrm{d}\theta} = \frac{1}{\pi}\int_0^{2\pi} \widehat{g}(\theta)\cos(n\theta)\,\mathrm{d}\theta$$

$$B_n = \frac{(\widehat{g}(\theta), \sin(n\theta))}{(\sin(n\theta), \sin(n\theta))} = \frac{\displaystyle\int_0^{2\pi} \sin(n\theta)\widehat{g}(\theta)\,\mathrm{d}\theta}{\displaystyle\int_0^{2\pi} \sin^2(n\theta)\,\mathrm{d}\theta} = \frac{1}{\pi}\int_0^{2\pi} \widehat{g}(\theta)\sin(n\theta)\,\mathrm{d}\theta$$

For the following examples, we will apply the *Principle of Superposition*.

Example 6.1.1

Solve the BVP

$$\begin{cases} \Delta u = 0 & 0 < r < 1 \\ u(1,\theta) = 1 + \sin(\theta) + \frac{1}{2}\sin(3\theta) + \cos(4\theta) & 0 \le \theta < 2\pi \end{cases}$$

Solution 1.

$A_0 = 1,\ B_1 = 1,\ B_3 = \frac{1}{2}, A_4 = 1$. So,

$$u(r,\theta) = 1 + r\sin(\theta) + \frac{1}{2}r^3\sin(3\theta) + r^4\cos(4\theta).$$

[*We applied the Principle of Superposition*] □

Example 6.1.2

Solve the BVP

$$\begin{cases} \Delta u = 0 & 0 < r < 1 \\ u(1,\theta) = 1 + \sin(\theta) + \frac{1}{a}\cos(\theta) & 0 \le \theta < 2\pi \end{cases}$$

Solution 2.

$A_0 = 1,\ B_1 = 1,\ A_1 = \frac{1}{2}$. So,

$$u(r,\theta) = 1 + r\sin(\theta) + \frac{1}{2}r\cos(\theta)$$

□

Example 6.1.3

Solve the BVP

$$\begin{cases} \Delta u = 0 & 0 < r < 1 \\ u(1, \theta) = 2 & 0 \leq \theta < 2\pi \end{cases}$$

Solution 3.

The boundary condition is independent of θ. So, our solution is constant and is also independent of θ. That is,

$$u(r, \theta) = 2.$$

□

Example 6.1.4

Solve the BVP

$$\begin{cases} \Delta u = 0 & 0 < r < 1 \\ u(1, \theta) = \sin(\theta) & 0 \leq \theta < 2\pi \end{cases}$$

Solution 4.

$B_1 = 1$. So,

$$u(r, \theta) = r \sin(\theta).$$

□

Example 6.1.5

Solve the BVP

$$\begin{cases} \Delta u = 0 & 0 < r < 1 \\ u(1, \theta) = \sin(3\pi) & 0 \leq \theta < 2\pi. \end{cases}$$

Solution 5.

$B_3 = 1$. So,

$$u(r, \theta) = r^3 \sin(3\theta).$$

□

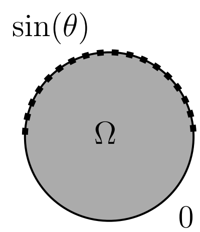
Sometimes, we can also have more complicated boundary condition that is piecewise defined.

Example 6.1.6

Solve the BVP

$$\begin{cases} \Delta u = 0 & 0 < r < 1 \\ u(1,\theta) & = \begin{cases} \sin\theta & 0 \le \theta < \pi \\ 0 & \pi \le \theta < 2\pi \end{cases} \end{cases}$$

Solution 6.



$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_0^{\pi} \sin(\theta)\,\mathrm{d}\theta + \frac{1}{2\pi}\int_{\pi}^{2\pi} 0\,\mathrm{d}\theta \\ A_n &= \frac{1}{\pi}\int_0^{\pi} \sin(\theta)\cos(n\theta)\,\mathrm{d}\theta + \frac{1}{\pi}\int_0^{\pi} 0\cdot\cos(n\theta)\,\mathrm{d}\theta \\ B_n &= \frac{1}{\pi}\int_0^{\pi} \sin(\theta)\sin(n\theta)\,\mathrm{d}\theta + \frac{1}{\pi}\int_0^{\pi} 0\cdot\sin(n\theta)\,\mathrm{d}\theta \end{aligned}$$

So,

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n[A_n\cos(n\theta) + B_n\sin(n\theta)].$$

□

6.2 Boundary Value Problems

Definition 6.2.1 (Steady-State Problems). Solution does not change in time.

Example 6.2.2 Steady-State Heat Equation

$$\begin{aligned} &\text{[PDE]} && \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \sin(\pi x), && 0 < x < 1, t > 0 \\ &\text{[BCs]} && u(t,0) = u(t,1) = 0 && t > 0 \\ &\text{[IC]} && u(0,x) = \sin(3\pi x) && 0 < x < 1 \end{aligned}$$

To find the steady-state solution, find $u(\infty, x)$. If $u(\infty, x)$ exists, we solve

$$\begin{cases} \dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \sin(\pi x) & 0 < x < 1 \\ u(0) = u(1) = 0. \end{cases}$$

So,

$$u(\infty, x) = \frac{1}{\pi^2}\sin(\pi x).$$

Definition 6.2.3 (Dirichlet Problems (First Kind of BC)). The PDE holds over a given region of space, and the solution is specified on the boundary of that region.

- Interior Dirichlet Problem

$$\begin{cases} \Delta u = 0 & 0 < r < 1 \\ u(1, \theta) = u_1(\theta) & 0 \le \theta < 2\pi. \end{cases}$$

- Exterior Dirichlet Problem

$$\begin{cases} \Delta u = 0 & r > 1 \\ u(1, \theta) = u_1(\theta) & 0 \le \theta < 2\pi. \end{cases}$$

Definition 6.2.4 (Neumann Problems (Second Kind of BC). The PDE holds on some region of space. The outward normal derivative $\partial u/\partial \mathbf{n}$ (proportional to the inward flux) is specified on the boundary.

$$\begin{cases} \Delta u = 0 & 0 < r < 1 \\ \dfrac{\partial u}{\partial r}(1, \theta) = v_1(\theta) & 0 \le \theta < 2\pi, \end{cases}$$

where $\dfrac{\partial u}{\partial r}$ is the normal derivative. We also have the following property:

$$\int_0^{2\pi} \frac{\partial u}{\partial r}\,\mathrm{d}\theta = 0.$$

That is, the temperature of each point inside the circle does not change with respect to time.

Remark.

- Neumann problem makes sense only if the net flux across the region is 0. i.e., only when
$$\int_0^{2\pi} \frac{\partial u}{\partial r}\,\mathrm{d}\theta = 0$$
- The solution to Neumann problems is *not* unique.

Example 6.2.5 Non-Unique Solution to Neumann Problem

$$\begin{cases} \Delta u = 0 & 0 < r < 1 \\ \dfrac{\partial u}{\partial r}(1,\theta) = \cos(2\theta) & 0 \le \theta < 2\pi \end{cases}$$

has infinitely many solutions. But they just differ by constant:

$$u(r,\theta) = r^2\cos(2\theta) + C.$$

Definition 6.2.6 (Robin Problems (Third Kind of BC)). The PDE is given in some region of space, but the condition on the boundary is a mixture of the first two kinds.

$$\frac{\partial u}{\partial \mathbf{n}} + h(u-g) = 0,$$

where

- $\dfrac{\partial u}{\partial \mathbf{n}}$ is the normal derivative
- h is a constant
- g is a given function on the boundary.

We can rewrite the condition into

$$\frac{\partial u}{\partial \mathbf{n}} = -h(u-g).$$

So, the inward flux across the boundary is proportional to the difference between the temperature u and some specified temperature g. This exactly reflects the Newton's Law of Cooling.

Example 6.2.7 Robin Problem

$$\begin{cases} \Delta u = 0 & 0 < r < 1 \\ \dfrac{\partial u}{\partial r}(1, \theta) = -h(u - \sin\theta) & 0 \le \theta < 2\pi. \end{cases}$$

So, here $g(\theta) = \sin\theta$.

- If $h = 0$: $\dfrac{\partial u}{\partial r} = 0$. Then, the solution is not unique and will be constant.
- If h gets larger, the solution will move like the solution to the Dirichlet problem with BC: $u = g(\theta) = \sin\theta$.
- If h is positive but close to zero, then the solution will be almost zero (the average of $g(\theta) = \sin\theta$ on the boundary).

6.3 More Complicated BCs

Example 6.3.1 A Ring

Find a formula for the solution of the following BVP:

$$\begin{cases} \Delta u = 0 & R_1 < r < R_2 \\ u(R_1, \theta) = g_1(\theta) & \\ u(R_2, \theta) = g_2(\theta) & 0 \le \theta < 2\pi. \end{cases}$$

Solution 1.

In polar coordinate:

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Using separation of variables, assume $u(r, \theta) = R(r)\Theta(\theta)$. Then,

$$① \quad r^2 R'' + rR' - \lambda^2 R = 0 \quad \text{and} \quad ② \quad \Theta'' + \lambda^2 \Theta = 0.$$

For ①:

- If $n = \lambda \neq 0$: $R_n(r) = c_1 r^n + c_2 r^{-n}$
- If $n = \lambda = 0$: $R_n(r) = c_3 + c_4 \ln r$.

For ②:

$$\Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta).$$

Since $R_1 < r < R_2$, nothing will cause trouble, and we should keep everything here.

$$\begin{aligned}
u(r,\theta) &= (c_3 + c_4 \ln r)A_0 + \sum_{n=1}^{\infty} \left(c_1 r^n + c_2 r^{-n}\right)[A_n \cos(n\theta) + B_n \sin(n\theta)] \\
&= k_0 + k_1 \ln r + \sum_{n=1}^{\infty} \left(D_n r^n + E_n r^{-n}\right) \cos(n\theta) + \left(F_n r^n + G_n r^{-n}\right) \sin(n\theta) \\
u(R_1,\theta) &= k_0 + k_1 \ln R_1 + \sum_{n=1}^{\infty} \left(D_n R_1^n + E_n R_1^{-n}\right) \cos(n\theta) + \left(F_n R_1^n + G_n R_1^{-n}\right) \sin(n\theta) = g_1(\theta) \\
u(R_2,\theta) &= k_0 + k_1 \ln R_2 + \sum_{n=1}^{\infty} \left(D_n R_2^n + E_n R_2^{-n}\right) \cos(n\theta) + \left(F_n R_2^n + G_n R_2^{-n}\right) \sin(n\theta) = g_2(\theta).
\end{aligned}$$

Apply Fourier Expansion on $g_1(\theta)$ and $g_2(\theta)$:

$$\begin{aligned}
g_1(\theta) &= A_{1,0} + \sum_{n=1}^{\infty} A_{1.n} \cos(n\theta) + B_{1,n} \sin(n\theta) \\
g_2(\theta) &= A_{2,0} + \sum_{n=1}^{\infty} A_{2,n} \cos(n\theta) + B_{2,n} \sin(n\theta).
\end{aligned}$$

Matching coefficients, we have two systems to solve:

$$\begin{cases} k_0 + k_1 \ln R_1 = A_{1,0} \\ D_n R_1^n + E_n R_1^{-n} = A_{1,n} \\ F_n R_1^n + G_n R_1^{-n} = B_{1,n} \end{cases} \quad \text{and} \quad \begin{cases} k_0 + k_1 \ln R_2 = A_{2,0} \\ D_n R_2^n + E_n R_2^{-n} = A_{2,n} \\ F_n R_2^n + G_n R_2^{-n} = B_{2,n} \end{cases}$$

□

Example 6.3.2 A Ring in Action

Solve the BVP:

$$\begin{cases} \Delta u = 0 & 1 < r < 2 \\ u(1,\theta) = 0 & \\ u(2,\theta) = \sin(\theta) & 0 \le \theta < 2\pi. \end{cases}$$

Solution 2.

Applying Fourier's Expansion:

$$g_1(\theta) = 0 \implies A_{1,0} = 0,\ A_{1,n} = 0,\ B_{1,n} = 0$$
$$g_2(\theta) = \sin\theta \implies A_{2,0} = 0,\ A_{2,n} = 0,\ B_{2,n,n\neq 1} = 0,\ B_{2,1} = 1.$$

Hence, we need to solve

$$\begin{cases} k_0 + k_1 \ln(1) = 0 \\ D_n 1^n + E_n 1^{-n} = 0 \\ F_n 1^n + G_n 1^{-n} = 0, \quad n \neq 1 \\ F_1 \cdot 1 + G_1 \cdot 1 = 0 \end{cases} \quad \text{and} \quad \begin{cases} k_0 + k_1 \ln(2) = 0 \\ D_n 2^n + E_n 2^{-n} = 0 \\ F_n 2^n + G_n 2^{-n} = 0 \quad n \neq 1 \\ F_1 \cdot 2 + G_1 \cdot 2 = 1. \end{cases}$$

□

Example 6.3.3 Exterior Dirichlet BC

Find a formula for the solution of the following BVP:

$$\begin{cases} \Delta u = 0 & 1 < r < \infty \\ u(1,\theta) = g(\theta) & 0 \leq \theta \leq 2\pi. \end{cases}$$

Solution 3.

The most general solution we can get from Laplace Equation is

$$u(r,\theta) = \sum_{n=1}^{\infty} \Theta(\theta) R_n(r),$$

where

- $\Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)$
- $R_0(r) = c_1 + c_2 \ln(r)$
- $R_n(r) = c_3 r^n + c_4 r^{-n}$.

Since we don't want irregularity (things that will blow up), we drop $\ln(r)$ and r^n. Hence,

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^{-n}[A_n \cos(n\theta) + B_n \sin(n\theta)].$$

□