IB Mathematics Analysis and Approaches HL

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Contents

1 Number and Algebra

1.1 Sequences and Series

1. Terms: $u_1,\ u_2,\ u_3 \ldots$
   Position: n
   Sum: S

2. **Arithmetic Sequence**/Arithmetic Progression (AP):

   - Recursive formula: $u_{n+1} = u_n + d$, d is the common difference.
   - Explicit formula: $u_n = u_1 + d(n-1)$
   - Summation: $S_n = \frac{n}{2}[2u_1 + d(n-1)]$

     Proof 1.1.1 Let $u_1,\ u_2,\ u_3,\ \ldots,\ u_n$ be an arithmetic sequence with d as common difference.
     Then, $S_n = u_1 + u_2 + u_3 + \ldots + u_n = u_1 + (u_1 + d) + (u_1 + 2d) + \ldots + (u_1 + (n-1)d)$
     Also, $S_n = [u_1 + (n-1)d] + \ldots + (u_1 + d) + u_1$.
     Add two expressions together:

     $$2S_n = [2u_1 + (n-1)d]n$$

     $$\therefore S_n = \frac{n}{2}[2u_1 + (n-1)d].$$

3. **Geometric Sequence**

   - Recursive formula: $u_{n+1} = r \cdot u_n$, r is the common ratio.
   - Explicit formula: $u_n = u_1 \cdot r^{n-1}$
   - $$r = \frac{u_2}{u_1} = \frac{u_3}{u_2} = \frac{u_4}{u_3} = \ldots$$
   - Summation: $S_n = \frac{u_1(r^n - 1)}{r-1}$

     Proof 1.1.2 Let $u_1,\ u_2,\ u_3,\ \ldots,\ u_n$ be a geometric sequence with r as common ratio.
     $S_n = u_1 + u_2 + u_3 + \ldots + u_n = u_1 + (u_1 \cdot r) + (u_1 \cdot r^2) + \ldots + (u_1 \cdot r^{n-1})$
     Then, $rS_n = (u_1 \cdot r) + (u_1 \cdot r^2) + \ldots + (u_1 \cdot r^n)$.
     Subtract the first expression from the second:

     $$rS_n - S_n = u_1 \cdot r^n - u_1 \Rightarrow (r-1)S_n = u_1(r^n - 1)$$

     $$\therefore S_n = \frac{u_1(r^n - 1)}{r-1}$$

   - If $r > 1$, the sequence is an exponential growth.
     If $0 < r < 1$, the sequence has an exponential decay.
   - When $r > 1$, series approaches ∞.
     When $-1 < r < 1$, or $|r| < 1$, the series converges:

     $$S_\infty = \frac{u_1}{1-r}, |r| < 1$$

1.2 Exponents and Logarithms

1. $a^m \cdot a^n = a^{m+n}$
   $a^m \div a^n = a^{m-n}$
   $(a^m)^n = a^{mn}$

2. $x^0 = 1$ $(x^0 = x^{1-1} = \frac{x^1}{x^1} = 1)$
   $x^{-m} = \frac{1}{x^m}$
   $x^{\frac{1}{n}} = \sqrt[n]{x}$ $(x^{\frac{m}{n}} = (\sqrt[n]{x})^m)$

3. If $a = b$, then $a^n = b^n$
   If $m = n$, then $a^m = a^n$
   For $a^b = 1$: $a = 1, b \in \mathbb{R}$; $a \neq 1, b = 0$; OR $a = -1, b = 2n$

4. When solving exponential equations, convert them to the same base.

5. Division Theorem.

   Theorem 1.2.1 If $a^x = b^y$ given $a > 0$ and $b > 0$, then $a = b^{\frac{y}{x}}$.

   Proof 1.2.1

$$a^x = b^y$$

$$(a^x)^{\frac{1}{x}} = (b^y)^{\frac{1}{x}} \Rightarrow a = b^{\frac{y}{x}}$$

6. $a = b^x \Leftrightarrow x = \log_b a$, where $a, b \in \mathbb{R}^+$ and $b \neq 1$.

7. Logarithmic rules:

   - $\log_a x + \log_a y = \log_a (xy)$

     Proof 1.2.2 Let $\log_a x = p,\ \log_a y = q.\ \Rightarrow a^p = x, a^q = y$.
     Then, $x \cdot y = a^p \cdot a^q = a^{p+q}$.

$$\therefore \log_a (xy) = p + q = \log_a x + \log_a y.$$

   - $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$

     Proof 1.2.3 Let $\log_a x = p,\ \log_a y = q.\ \Rightarrow a^p = x, a^q = y$.
     Then, $\frac{x}{y} = \frac{a^p}{a^q} = a^{p-q}$.

$$\therefore \log_a \left(\frac{x}{y}\right) = p - q = \log_a x - \log_a y.$$

   - $\log_a x^n = n \log_a x$
   - $\log_a 1 = 0$
   - $\log_a a = 1$

- $-\log_a x = \log_a \frac{1}{x}$
- $\log_a x = \frac{\log_b x}{\log_b a}$
- $\log_a b = \frac{1}{\log_b a}$

1.3 Proof

1. Direct proof:

   Example 1.3.1 Show that the sum of two even numbers is always even.
   Let m and n be two even positive integers.
   $m = 2p, n = 2q$, where p and $q \in \mathbb{Z}^+$.
   Then, $m+n = 2p+2q = 2(p+q)$, which is an even number.

   Example 1.3.2 Show that $\left(x+\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 \equiv x^2 + ax$.

   $$\text{LHS} = x^2 + \frac{a^4}{4} + ax - \frac{a^4}{4} = x^2 + ax = \text{RHS}.$$

   Equations "=": only true from some values.
   Identities "≡": true for all values.

   Example 1.3.3 Prove that if the sum of the digits of a four-digit number is divisible by 3, then the four-digit number is also divisible by 3.

   Example 1.3.4 Let n be a 4-digit number: $n = 1000a + 100b + 10c + d$, where $0 \le a,b,c,d \le 9$, and $a \neq 0$.
   It is given that $a+b+c+d = 3k, k \in \mathbb{Z}$:

   $$\begin{aligned} n &= 1000a + 100b + 10c + d + 3k - a - b - c - d \\ &= 999a + 99b + 9c + 3k \\ &= 3(333a + 33b + 3c + k) \end{aligned}$$

   Since $(333a + 33b + 3c + k) \in \mathbb{Z}$, it implies that n is divisible by 3.

2. Proof by Contradiction:

   Example 1.3.5 Prove the statement: If the integer n is odd, then n^2 is also odd.
   Let, if possible, n^2 is even and n is odd.
   Then, $n^2 = 2k,\ k \in \mathbb{Z} \Rightarrow n \times n = 2k$, which indicates the product of two odd number is even, and which is not true.
   Hence, there is a contradiction.
   $\therefore$ Our assumption is wrong, and thus given that n is odd, n^2 is also odd.

   Example 1.3.6 Show that $\sqrt{2}$ is irrational.
   Let us assume, if possible, that $\sqrt{2}$ is rational:
   $\sqrt{2} = \frac{p}{q}$, where $p, q \in \mathbb{Z}$, and p, q have no common factors, $q \neq 0$.
   $\therefore 2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2 \qquad (1).$

$\therefore p^2$ is even, and thus p is also even.
As p is an even number, we can write: $p = 2k, k \in \mathbb{Z}. \Rightarrow \therefore p^2 = (2k)^2 = 4k^2 \quad (2)$.
From (1) and (2): $4k^2 = 2q^2 \Rightarrow q^2 = 2k^2 \Rightarrow q^2$ is even, and thus q is also an even number.
But since p and q have no common factors, they cannot have "2" as a common factor. Hence, we have arrived at a contradiction.
$\therefore$ Our assumption is incorrect, and $\sqrt{(2)}$ is irrational.

Definition 1.3.1 A number is **rational** if it can be written as $\frac{p}{q}$, where $p, q \in \mathbb{Z}$, and $q \neq 0$.

Example 1.3.7 Prove that there is no $x \in \mathbb{R}$ such that $\frac{1}{x-2} = 1 - x$ Assume there is a real number x such that $\frac{1}{x-2} = 1 - x$.
$\therefore (1-x)(x-2) = 1 \Rightarrow x^2 - 3x + 3 = 0$
Solving the equation, we get $x = \frac{3 \pm \sqrt{9-12}}{2}$, which $\notin \mathbb{R}$
$\therefore$ We arrived at a contradiction, and our assumption is incorrect. There is no $x \in \mathbb{R}$ such that $\frac{1}{x-2} = 1 - x$

3. Proof by Mathematical Induction

Definition 1.3.2 Principle of Mathematical Induction (PMI):
Suppose P_n is a proposition which is defined for every integer $n \geq a,\ a \in \mathbb{Z}$. If P_a is true, and if P_{k+1} is true whenever P_k is true, then P_n is true $\forall n \geq a$.

Example 1.3.8 Prove that $4^n + 2$ is divisible by 3 for $n \in \mathbb{Z},\ n \geq 0$, by using PMI.
For $n = 0$, LHS $= 4^0 + 2 = 1 + 2 = 3$, which is divisible by 3.
$\therefore P_0$ (OR denoted as $P(0)$) is true.
Assume that P_k is true: i.e., $4^k + 2$ is divisible by 3. $\Rightarrow 4^k + 2 = 3A,\ A \in \mathbb{Z}^+ \Rightarrow 4^k = 3A - 2$.
Consider P_{k+1}:

$$\begin{aligned} 4^{k+1} + 2 &= 4^k \cdot 4^1 + 2 \\ &= (3A-2) \cdot 4 + 2 \\ &= 12A - 6 \\ &= 3(4A-2). \end{aligned}$$

$\therefore 4A - 2$ is an integer as $A \in \mathbb{Z}^+$, $4^{k+1} + 2$ is divisible by 3 whenever $4^k + 2$ is divisible by 3.
Since P_0 is true, and P_{k+1} is true whenever Pk is true, P_n is true $\forall n \in \mathbb{Z},\ n \geq 0$.

Example 1.3.9 A sequence is defined by $u_{n+1} = 2u_n + 1\ \forall n \in \mathbb{Z}^+$. Prove that $u_n = 2^n - 1$.
For $n = 1,\ u_1 = 2^1 - 1 = 1 \Rightarrow \therefore P_1$ is true.
Let P_k be true: $u_k = 2^k - 1$ for some $k \in \mathbb{Z}^+$.
Consider P_{k+1}:

$$\begin{aligned} u_{k+1} &= 2u_k + 1 \\ &= 2(2^k - 1) + 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true, P_n is true $\forall n \in \mathbb{Z}^+$.

Example 1.3.10 Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6},\ \forall n \in \mathbb{Z}^+$.
For $n = 1$, LHS $= 1^2 = 1$, RHS $= \frac{1(1+1)(2+1)}{6} = 1$

$\therefore$ LHS $=$ RHS $\Rightarrow P_1$ is true.
Assume that P_k is true, $k \in \mathbb{Z}^+$: $1^2+2^2+3^2+\cdots+k^2=\frac{k(k+1)(2k+1)}{6}$.
Consider P_{k+1}:

$$\begin{aligned}
\text{LHS} &= 1^2+2^2+3^2+\cdots+k^2+(k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6}+(k+1)^2 \\
&= \frac{k(k+1)(2k+1)+6(k+1)^2}{6} \\
&= \frac{(k+1)[k(2k+1)+6(k+1)]}{6} \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6} = \text{RHS}.
\end{aligned}$$

Thus, P_{k+1} is true whenever P_k is true.
Since P_1 is true, and P_{k+1} is true whenever P_k is true, P_n is true $\forall n \in \mathbb{Z}^+$.

Example 1.3.11 Prove that if $x \neq 1$, the $\prod_{i=1}^{n}(1+x^{2^{i-1}})=(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^{n-1}})=\frac{1-x^{2^n}}{1-x}$.

For $n=1$, LHS $=1+x$, RHS $=\frac{1-x^{2^1}}{1-x}=\frac{1-x^2}{1-x}=1+x$. $\Rightarrow \therefore$ LHS $=$ RHS, P_1 is true.
Assume that P_k is true: $(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^{k-1}})=\frac{1-x^{2^k}}{1-x}$.
Consider P_{k+1}:

$$\begin{aligned}
\text{LHS} &= (1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^{k-1}})(1+x^{2^k}) \\
&= \frac{1-x^{2^k}}{1-x}(1+x^{2^k}) \\
&= \frac{1+x^{2^k}-x^{2^k}+(x^{2^k})^2}{1-x} \\
&= \frac{1-x^{2^k\cdot 2}}{1-x} \\
&= \frac{1-x^{2^{k+1}}}{1-x} = \text{RHS}.
\end{aligned}$$

Since P_1 is true, and P_{k+1} is true whenever P_k is true, P_n is true $\forall n \in \mathbb{Z}^+$.

1.4 Counting and Binomial Theorem

1. Choose r from n: $\binom{n}{r} = {}_nC_r$
   - $\binom{n}{m}=\binom{n}{n-m}$
   - $\binom{n}{r}=\frac{n!}{r!(n-r)!}$
   - Factorial notation: $n! = n(n-1)(n-2)\cdots 2\cdot 1$
     e.g. $\binom{5}{3}=\frac{5!}{3!(5-3)!}=\frac{5\times4\times3!}{3!\times2}=5\times2=10.$

Example 1.4.1 **Write $\frac{(n!)^2}{(n-1)!(n-2)!}$ without using factorial notation.**

$$(n!)^2 = n! \times n! = n(n-1)! \times n(n-1)(n-2)!$$

$$\therefore \frac{(n!)^2}{(n-1)!(n-2)!} = \frac{n(n-1)! \times n(n-1)(n-2)!}{(n-1)!(n-2)!} = n \cdot n(n-1) = n^3 - n^2.$$

2. The number of ways of arranging n distinct objects in a row is $n!$.

3. The number of permutations of r objects out of n distinct objects is given by

$$_nP_r = \frac{n!}{(n-r)!}.$$

4. In permutations, the order matters.
   In combinations, the order does not matter.

5. The Binomial Theorem:

   Theorem 1.4.1

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + b^n,\ n \in \mathbb{N}$$
$$= \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

Example 1.4.2 **Find $(2x+3)^4$.**

$$(2x+3)^4 = (2x)^4 + \binom{4}{1}(2x)^3(3)^1 + \binom{4}{2}(2x)^2(3)^2 + \binom{4}{3}(2x)(3)^3 + 3^4$$
$$= 16x^4 + 96x^3 + 216x^2 + 216x + 81$$

Example 1.4.3 **Find the term independent of x in the expansion of $\left(x - \frac{2}{x^2}\right)^{12}$.**

General term: $\binom{12}{r}x^{12-r}\left(-\frac{2}{x^2}\right)^r$

Thus, the general expression for $x: x^{12-r-2r} = x^{12-3r}$

When $12-3r=0$, the term is independent of x: $12-3r=0 \Rightarrow r=4$.

$$\therefore \binom{12}{4}x^{12-4}\left(-\frac{2}{x^2}\right)^4 = 7920.$$

1. The independent term should not involve x in it since the independent term does not vary as x varies. (constant term)

2. The coefficient should not include x as well.

Example 1.4.4 **Find the coefficient of x^3y^2 in the expansion of $(2x+y)\left(x+\frac{y}{x}\right)^5$.**

Assume $2x \cdot A$ and $y \cdot B$ will yield the term x^3y^2. $\Rightarrow A = x^2y^2,\ B = x^3y$.

General term: $\binom{5}{r}x^{5-r}(\frac{y}{x})^r = \binom{5}{r}x^{5-2r}y^r$.
When $r=2,\ 5-2r=1\neq 2 \Rightarrow x^2y^2$ is not possible.
When $r=1,\ 5-2r=3 \Rightarrow x^3y$ is possible.

$$\therefore \text{Coefficient} = \binom{5}{1} = 5.$$

Example 1.4.5 Find the coefficient of x^2 in the expansion of $(1-2x)(1-4x)^7$.
Assume $1\cdot A = x^2,\ -2x\cdot B = x^2. \Rightarrow A = x^2,\ B = x.$
General term: $\binom{7}{r}(-4x)^{7-r}(1)^r$
When $7-r=2,\ r=5:\ \binom{7}{5}(-4x)^2(1)^5 = 336x^2. \Rightarrow 1\cdot 336x^2 = 336x^2$
When $7-r=1,\ r=6:\ \binom{7}{6}(-4x)^1(1)^6 = -28x. \Rightarrow (-2x)\cdot(-28x) = 56x^2$

$$\therefore \text{Coefficient} = 336+56 = 392.$$

6. AHL - Extension of Binomial Theorem:

Theorem 1.4.2

$$\begin{aligned}(a+b)^n &= a^n\left(1+\frac{b}{a}\right)^n \\ &= a^n(1+n\cdot\frac{b}{a}+\frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2+\frac{n(n-1)(n-2)}{3!})\left(\frac{b}{a}\right)^3+\cdots),\ n\in\mathbb{Q},\ \left|\frac{b}{a}\right|<1\end{aligned}$$

Example 1.4.6 Expand $\sqrt{1+2x}$ $\left(|x|<\frac{1}{2}\right)$ and $\frac{2}{1-3x}$ $\left(|x|<\frac{1}{3}\right)$ up to x^3 term.

$$\begin{aligned}(1+2x)^{\frac{1}{2}} &= 1+\frac{1}{2}(2x)+\frac{1}{2}\left(\frac{1}{2}-1\right)\frac{(2x)^2}{2!}+\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\frac{(2x)^3}{3!}+\cdots \\ &= 1+x-\frac{1}{2}x^2+\frac{1}{2}x^3+\cdots.\end{aligned}$$

$$\begin{aligned}2(1-3x)^{-1} &= 2(1-(-3x)-(-1-1)\frac{(-3x)^2}{2!}-(-1-1)(-1-2)\frac{(-3x)^3}{3!}+\cdots \\ &= 2(1+3x+x^2+27x^3+\cdots \\ &= 2+6x+18x^2+54x^3+\cdots.\end{aligned}$$

Example 1.4.7 Write the first three terms in the expansion of $(2+x)^{-3}$.

$$\begin{aligned}(2+x)^{-3} &= 2^{-3}\left(1+\frac{x}{2}\right)^{-3} \\ &= \frac{1}{8}\left(1+(-3)\frac{x}{2}+(-3)(-3-1)\frac{2^2}{2\cdot 2!}+\cdots\right) \\ &= \frac{1}{8}\left(1-\frac{3}{2}x+\frac{12}{4}x^2+\cdots\right) \\ &= \frac{1}{8}-\frac{3}{16}x+\frac{3}{8}x^2+\cdots.\end{aligned}$$

Example 1.4.8 Find square root of 24 correct to 5 decimal places, using the binomial theorem.

$$
\begin{aligned}
24^{\frac{1}{2}} &= (25-1)^{\frac{1}{2}} = 25^{\frac{1}{2}}\left(1-\frac{1}{25}\right)^{\frac{1}{2}} \\
&= 5\left(1+\left(\frac{1}{2}\right)\left(-\frac{1}{25}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{25}\right)^2+\frac{\frac{1}{2\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}}{3!}\left(-\frac{1}{25}\right)^3+\cdots\right) \\
&= 5\left(1-\frac{1}{50}-\frac{1}{5000}-\frac{1}{250000}+\cdots\right) \\
&= 5(1-0.02-0.0002-0.000004) \\
&= 4.89898 \quad (5\ d.p.).
\end{aligned}
$$

1.5 Partial Fraction

1. Proper fractions: The degree of the numerator is less than the degree of the denominator.

2. Partial fraction: A method to separate one complex fraction into two or simpler fractions.

Example 1.5.1 Find the partial fraction of $\frac{3x}{(x-1)(x+2)}$.
Let $\frac{3x}{(x-1)(x+2)} = \frac{A}{x-1}+\frac{B}{x+2}$.

$$\therefore 3x \equiv A(x+2)+B(x-1).$$

When $x=1$, $3=3A \;\Rightarrow\; A=1$.
When $x=-2$, $-6=-3B \;\Rightarrow\; B=2$.

$$\therefore \frac{3x}{(x-1)(x+2)} \equiv \frac{1}{x-1}+\frac{2}{x+2}.$$

Example 1.5.2 Find the partial fraction of $\frac{2x+5}{(x-2)(x+1)}$.
Let $\frac{2x+5}{(x-2)(x+1)} = \frac{A}{x-2}+\frac{B}{x+1}$.

$$\therefore 2x+5 \equiv A(x+1)+B(x-2).$$

When $x=2$, $9=3A \;\Rightarrow\; A=3$.
When $x=-1$, $3=-3B \;\Rightarrow\; B=-1$.

$$\therefore \frac{2x+5}{(x-2)(x+1)} \equiv \frac{3}{x-2}-\frac{1}{x+1}.$$

Example 1.5.3 Find the partial fraction of $\frac{34-12x}{3x^2-10x-8}$.
As $\frac{34-12x}{3x^2-10x-8} = \frac{34-12x}{(3x+2)(x-4)}$, let $\frac{34-12x}{(3x+2)(x-4)} = \frac{A}{3x+2}+\frac{B}{x-4}$.

$$\therefore 34-12x \equiv A(x-4)+B(3x+2).$$

When $x=4$, $-14=14A \;\Rightarrow\; B=-1$.

When $x = -\frac{2}{3}$, $42 = -\frac{14}{3}A \Rightarrow A = -9$.

$$\therefore \frac{34-12x}{(3x+2)(x-4)} \equiv -\frac{9}{3x+2} - \frac{1}{x-4}.$$

1.6 Complex Number

1.6.1 Introduction

1. Complex Number:

   Definition 1.6.1 **Complex Numbers** are numbers in the form of $a+b\mathrm{i}$, where $\mathrm{i}^2 = -1$.

   - a is called the **real part**, denoted as $\mathrm{Re}(a+b\mathrm{i}) = a$.
   - b is called the **imaginary part**, denoted as $\mathrm{Im}(a+b\mathrm{i}) = b$.

   $a+b\mathrm{i}$ is called the Cartesian form of complex number.

2. Basic Calculations of Complex Number:

   - Define $z_1 = a+b\mathrm{i}$ and $z_2 = c+d\mathrm{i}$:

   $$z_1 \pm z_2 = (a \pm c) + (b \pm d)\mathrm{i}.$$

   - Define $z_1 = a+b\mathrm{i}$ and $z_2 = c+d\mathrm{i}$:

   $$z_1 z_2 = (ac-bd) + (ad+bc)\mathrm{i}.$$

   Proof 1.6.1

   $$\begin{aligned} z_1 z_2 &= (a+b\mathrm{i})(c+d\mathrm{i}) \\ &= ac + (ad+bc)\mathrm{i} + bd\mathrm{i}^2 \qquad [\mathrm{i}^2 = -1] \\ &= (ac-bd) + (ad+bc)\mathrm{i}. \end{aligned}$$

   - Conjugate complex number:

   Definition 1.6.2 We call $a-b\mathrm{i}$ as the **conjugate** of $z = a+b\mathrm{i}$, denoted as $z^* = a-b\mathrm{i}$.

   Theorem 1.6.1 Define $z_1 = a+b\mathrm{i}$, and z^* is the conjugate of z_1. Then,

   $$z_1 z^* = a^2 + b^2.$$

   Proof 1.6.2 By definition, $z^* = a-b\mathrm{i}$. Thus,

   $$\begin{aligned} z_1 z^* &= (a+b\mathrm{i})(a-b\mathrm{i}) \\ &= a^2 - (b\mathrm{i})^2 \\ &= a^2 + b^2. \end{aligned}$$

- Define $z_1 = a + bi$ and $z_2 = c + di$:

$$\frac{z_1}{z_2} = \frac{ac+bd}{c^2+d^2} - \frac{bc-ad}{c^2+d^2}\mathrm{i}.$$

Proof 1.6.3

$$\begin{aligned}\frac{z_1}{z_2} = \frac{a+b\mathrm{i}}{c+d\mathrm{i}} &= \frac{(a+b\mathrm{i})(c-d\mathrm{i})}{(c+d\mathrm{i})(c-d\mathrm{i})} \\ &= \frac{(ac+bd)-(bc-ad)\mathrm{i}}{c^2+d^2} \\ &= \frac{ac+bd}{c^2+d^2} - \frac{bc-ad}{c^2+d^2}\mathrm{i}.\end{aligned}$$

Example 1.6.1 Find $z \in \mathbb{C}$ that satisfies the equation $\frac{z+2}{1-\mathrm{i}} = \frac{z-3\mathrm{i}}{2+\mathrm{i}}$.

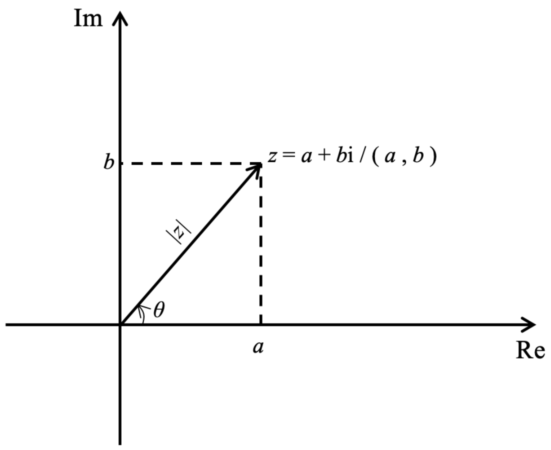
$$\begin{aligned}(z+2)(2+\mathrm{i}) &= (z-3\mathrm{i})(1-\mathrm{i}) \\ z(2+\mathrm{i})+4+2\mathrm{i} &= z(1-\mathrm{i})-3\mathrm{i}+(3\mathrm{i})^2 \\ z(2+\mathrm{i}-1+\mathrm{i}) &= -3\mathrm{i}-3-4-2\mathrm{i} \\ z(1+2\mathrm{i}) &= -7-5\mathrm{i} \\ z &= \frac{-7-5\mathrm{i}}{1+2\mathrm{i}} = -\frac{17}{5}+\frac{9}{5}\mathrm{i}.\end{aligned}$$

3. If $s = a + bi$ and $t = c + di$, then:

$$\mathrm{Re}(s)+\mathrm{Re}(t) = \mathrm{Re}(s+t); \text{ and } \mathrm{Im}(\mathrm{i}\cdot s) = \mathrm{Re}(s).$$

1.6.2 Argand Diagram

1. The Complex Plane:



$z = a + bi$ can be represented on a complex plane with real coordinate a and imaginary coordinate b. It can also be denoted as $z(a,b)$.

- Modulus of a complex number:
$$|z| = \sqrt{a^2 + b^2}.$$
- Argument of a complex number:
$$\text{Arg}(z) = \arctan\left(\frac{b}{a}\right)(+k\pi) \rightarrow \arctan x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

  *When determine a complex number, first draw it on the plane to show which quadrant it is in.
  The range of argument is $[0, 2\pi]$ or $[-\pi, \pi]$.
- Use modulus and argument to express a complex number:
$$a = |z| \cdot \cos\theta;$$
$$b = |z| \cdot \sin\theta.$$

2. If $z = a + b\text{i}$ and $|z| = 1$, then $z^* = z^{-1}$.

Proof 1.6.4

$$\because |z| = 1$$
$$\therefore \sqrt{a^2 + b^2} = 1$$
$$\therefore a^2 + b^2 = 1$$

Method 1

$$\begin{aligned}\text{RHS} = z^{-1} &= \frac{1}{a + b\text{i}} = \frac{a - b\text{i}}{(a + b\text{i})(a - b\text{i})} \\ &= \frac{a - b\text{i}}{a^2 + b^2} = a - b\text{i} \\ &= z^* = \text{LHS}.\end{aligned}$$

Method 2

$$\begin{aligned}z \cdot z^* &= (a + b\text{i})(a - b\text{i}) \\ &= a^2 + b^2 \\ &= |z|^2 = 1 \\ \therefore\ z^* &= z^{-1}\end{aligned}$$

3. When $|z| \neq 1$, $z^* = \frac{|z|^2}{z}$, and $z^{-1} = \frac{z^*}{|z|^2}$.

4. Properties of modulus and arguments:
   For complex number s and $t \in \mathbb{C}$:

   - $$|st| = |s||t|$$
   - $$\left|\frac{s}{t}\right| = \frac{|s|}{|t|}$$
   - $$\text{Arg}(st) = \text{Arg}(s) + \text{Arg}(t) + 2k\pi$$
   - $$\text{Arg}\left(\frac{s}{t}\right) = \text{Arg}(s) - \text{Arg}(t) + 2k\pi$$

1.6.3 Complex Number in Other Forms

1. The Polar Form (Modulus-Argument Form):

   - $$z = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{cis}\theta$$

     Proof 1.6.5 According to the Argand Diagram:

     $$z = x + y\mathrm{i} = r\cos\theta + \mathrm{i}r\sin\theta = r(\cos\theta + \mathrm{i}\sin\theta).$$

   - $$z_1 z_2 = r_1 r_2 \mathrm{cis}(\theta_1 + \theta_2)$$

   - $$\frac{z_1}{z_2} = \frac{r_1}{r_2}\mathrm{cis}(\theta_1 - \theta_2)$$

2. de Movrie's Theorem:

   - By Maclaurin Series:
     $$e^{\mathrm{i}\theta} = \mathrm{cis}\theta = \cos\theta + \mathrm{i}\sin\theta.$$

   - Exponential form of complex number:

     $$z = re^{\mathrm{i}\theta} = r\mathrm{cis}\theta.$$

3. Cartesian Form: Addition and Subtraction
   Modulus-Argument Form: Multiply and Division
   Exponential Form: Exponents and Roots

4. Since $\mathrm{cis}\theta = \mathrm{cis}(\theta + 2k\pi)$,
   $$re^{\mathrm{i}\theta} = re^{\mathrm{i}(\theta + 2k\pi)}.$$

   Example 1.6.2 Find $e^{\mathrm{i}\frac{17\pi}{12}}$ in the form of Cartesian.

   $$\begin{aligned} e^{\mathrm{i}\frac{17\pi}{12}} = e^{\mathrm{i}\left(\frac{7\pi}{6} + \frac{\pi}{4}\right)} &= e^{\mathrm{i}\frac{7\pi}{6}} \cdot e^{\mathrm{i}\frac{\pi}{4}} \\ &= \mathrm{cis}\left(\frac{7\pi}{6}\right) \cdot \mathrm{cis}\left(\frac{\pi}{4}\right) \\ &= \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\mathrm{i}\right)\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}\right) = \frac{\sqrt{2} - \sqrt{6}}{4} - \frac{\sqrt{2} + \sqrt{6}}{4}\mathrm{i}. \end{aligned}$$

1.6.4 Power of Complex Number

1. For a complex number $z = re^{\mathrm{i}\theta}$,
   $$z^n = r^n e^{\mathrm{i}n\theta}.$$

Example 1.6.3 Find $\left(3\cos\frac{2\pi}{3} - 3\mathrm{i}\sin\frac{\pi}{3}\right)^3$

$$\begin{aligned}\left(3\cos\frac{2\pi}{3} - 3\mathrm{i}\sin\frac{\pi}{3}\right)^3 &= \left(-3\cos\frac{\pi}{3} - 3\mathrm{i}\sin\frac{\pi}{3}\right)^3 \\ &= \left(-3\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)\right)^3 \\ &= (-3)^3(e^{\mathrm{i}\frac{\pi}{3}})^3 \\ &= -27e^{\mathrm{i}\pi} \\ &= -27(-1) = 27.\end{aligned}$$

Key learnings:
1. $z = 3$ is only the fundamental root of equation $z^3 = 27$. In $\mathbb{C}$, there are other two complex roots that satisfy the equation.
2. In $\mathbb{C}$, $\sqrt{4} = \pm 2 = 2 + 0 \cdot \mathrm{i}$ or $-2 + 0 \cdot \mathrm{i}$.

Example 1.6.4 Given a complex number $\omega \neq 1$ **is one of the solutions of** $z^3 = 1$.
a. Prove $\omega^2 + \omega + 1 = 0$;
b. Calculate $\omega^{2019} + \omega^{2020} + \omega^{2021} + \omega^{2022}$.

(a) Approach A

$$\begin{aligned}&\because \omega^3 = 1 \\ &\therefore \omega^3 - 1 = 0 \;\Rightarrow\; (\omega - 1)(\omega^2 + \omega + 1) = 0 \\ &\because \omega \neq 1 \\ &\therefore \omega^2 + \omega + 1 = 0.\end{aligned}$$

Approach B $\omega^2 + \omega + 1 = 0$ is a geometric sequence, $u_1 = 1,\ r = \omega$:

$$S_3 = \frac{u_1(1 - r^3)}{1 - r} = \frac{1 - \omega^3}{1 - \omega} = \frac{0}{1 - \omega} = 0.$$

(b)

$$\begin{aligned}\omega^{2019} + \omega^{2020} + \omega^{2021} + \omega^{2022} &= \omega^{2019} \times (1 + \omega + \omega^2 + \omega^3) \\ &= \omega^{2019}(0 + 1) = \omega^{2019} \\ &= \left(\omega^3\right)^{673} = 1.\end{aligned}$$

Example 1.6.5 Find:
a. 1^{i};
b. $\ln(-1)$;
c. $\ln(-c)$, **where** c **is a constant.**

(a)

$$1 = e^{\mathrm{i}2\pi} \;\Rightarrow\; 1^{\mathrm{i}} = \left(e^{\mathrm{i}2\pi}\right)^{\mathrm{i}} = e^{-2\pi}. \quad (1^{\mathrm{i}} = e^{-2k\pi}, k \in \mathbb{Z})$$

(b)

$$-1 = e^{\mathrm{i}\pi} \;\Rightarrow\; \ln(-1) = \ln\left(e^{\mathrm{i}\pi}\right) = \mathrm{i}\pi.$$

(c)

$$\ln(-c) = \ln[(-1)\cdot c] = \ln(-1) + \ln(c) = \ln(c) + \mathrm{i}\pi.$$

1.6.5 Polynomial Function with Complex Roots

1. Conjugate Pair Theorem:

   Theorem 1.6.2 If z is a complex root of $P(x)$, then the conjugate of $z(z^*)$ is also a complex root of $P(x)$. ($P(x)$ should be a polynomial with rational coefficients.)

2. Properties of Conjugate.

   - $$(s \pm t)^* = s^* \pm t^*$$
   - $$(st)^* = s^* t^*$$
   - $$\left(\frac{s}{t}\right)^* = \frac{s^*}{t^*}$$

1.6.6 Root of Complex Numbers

1. The Root of Unity:

   Theorem 1.6.3 For any complex equation $\omega^n = 1$, there are n distinct roots:

   $$1 = e^{\mathrm{i}(0+2k\pi)} = \omega^n,\ k \in \mathbb{Z} \quad \Rightarrow \quad \omega = e^{\mathrm{i}\frac{2k\pi}{n}},\ k \in \mathbb{Z}.$$

   Example 1.6.6 Solve $z^3 = 8$.

   $$\begin{aligned} z^3 = 8\cdot 1 = 8e^{i(0+2k\pi)} \ \Rightarrow\ z = 2e^{i\frac{2k\pi}{3}},\ k \in \mathbb{Z} \\ k = 0:\ z = 2 \\ k = 1:\ z = 2e^{i\frac{2\pi}{3}} = 2\mathrm{cis}\left(\frac{2\pi}{3}\right) = -1 + \sqrt{3}\mathrm{i} \\ k = 2:\ z = 2e^{i\frac{4\pi}{3}} = 2\mathrm{cis}\left(\frac{4\pi}{3}\right) = -1 - \sqrt{3}\mathrm{i} \end{aligned}$$

2. Property of cisθ:

   $$\mathrm{cis}(-\theta) = \cos\theta - \mathrm{i}\sin\theta$$

   Proof 1.6.6

   $$\begin{aligned} \cos\theta - \mathrm{i}\sin\theta &= \cos(-\theta) + \mathrm{i}\sin(-\theta) \\ &= \mathrm{cis}(-\theta). \end{aligned}$$

2 Functions

2.1 Foundations of Functions

1. Relations and functions:

   Definition 2.1.1 A **relation** R is a set of ordered pairs (x, y) such that $x \in A,\ y \in B$, and sets $A,\ B$ are not empty.

   Definition 2.1.2 A **function** f is a relation in which every x-value has a unique y-value.

2. Domain and Range:

   Definition 2.1.3 **Domain** is the set of x-values.

   Definition 2.1.4 **Range** is the set of y-values.

   - Domain and Range should be in interval notation.
     (a) Using intervals to express the inequalities

     Example 2.1.1
     $$[3,4[\text{ means } 3 \leq x < 4$$

     (b) If the interval will be joint, we use $\cup$ to join the interval.

     Example 2.1.2
     $$3 < x < 4 \text{ or } x \geq 5 \ \Rightarrow\ ]3,4[\,\cup\,[5,+\infty[$$

     Example 2.1.3 Find the interval notation for the domain of $f(x) = \frac{1}{x}$.

     $$x \in\ ]-\infty,0[\,\cup\,]0,+\infty[\ \text{ OR } x \in \mathbb{R}\setminus 0$$

     Note: $\setminus$ means "exclude."

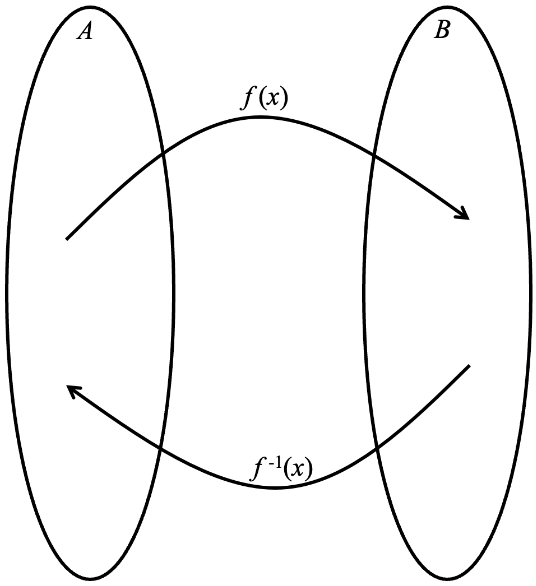
   - Since the y-values (outputs) depend on the x-values (inputs), y is the **dependent variable**, and x is the **independent variable**.
   - The independent variable x is also called the **argument** of the function.

3. Vertical Line test:

   - To test whether a relation is a function.
   - Since every x has one and only one value of y, there should be only one intersects.

4. Inverse of a function:

   Definition 2.1.5 $f^{-1}(x)$ is the **inverse function** of $f(x)$.



Example 2.1.4 $f(1)=3 \Rightarrow f^{-1}(3)=1$; $f(x)=x+5 \Rightarrow f^{-1}(x)=x-5$

- In inverse function, the input becomes the output, the output becomes the input.
- In inverse function, the domain becomes the range, the range becomes the domain.

Example 2.1.5 (a) **Find the inverse function of** $y=\dfrac{x+2}{3}$**.**

$$3y=x+2 \Rightarrow x=3y-2$$
$$f^{-1}(x)=3x-2$$

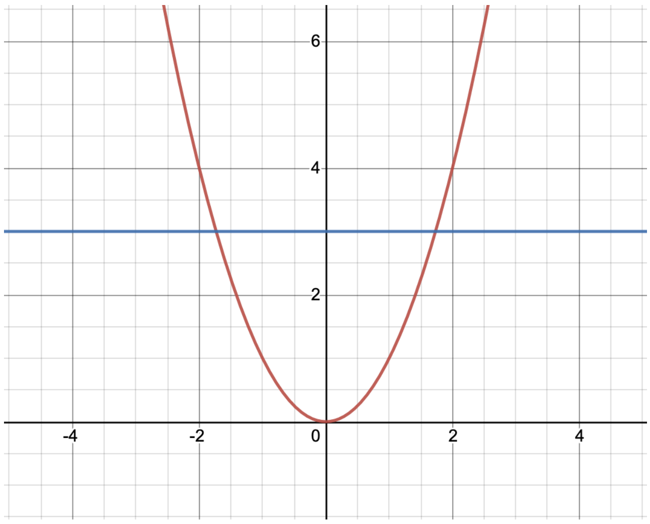
(b) **Find the inverse function of** $f(x)=\dfrac{x}{x+1}$**.**

$$y=\frac{x}{x+1} \Rightarrow xy+y=x \Rightarrow xy+x=y$$
$$\therefore y(x-1)=-x \Rightarrow y=-\frac{x}{x-1}$$

(c) **Find the inverse of** $\{.(4,2),(0,2),(-2,2)\}$

Inverse: $\{(2,4),(2,0),(2,-2)\}$

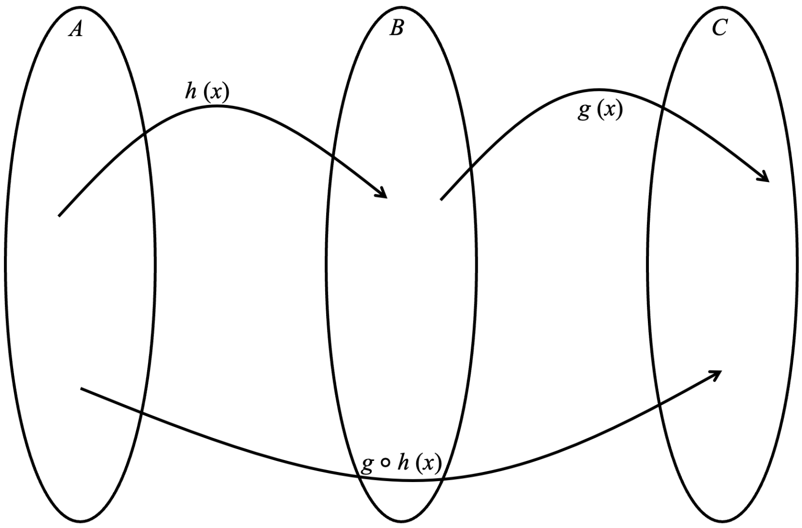
- By restricting the domain, we can find $f^{-1}(x)$ of $f(x)$, if the direct inverse of $f(x)$ is not a function.



Example 2.1.6 Horizontal line test: The largest domain we can find $f^{-1}(x)$ is $x \leq 0$ or $x > 0$.

5. Composite Functions:

Definition 2.1.6 We use $(g \circ h)(x)$ or $g(h(x))$ to represent composite functions.



Example 2.1.7 **Given** $f : x \mapsto 3x - 6,\ g : x \mapsto \frac{1}{3}x + 2$. **Find** $(f \circ g)(x)$ **and** $(g \circ f)(x)$**.**

$$(f \circ g)(x) = f(g(x)) = 3(\frac{1}{3}x + 2) - 6 = x.$$

$$(g \circ f)(x) = g(f(x)) = \frac{1}{3}(3x - 6) + 2 = x.$$

When f and g are inverse functions:

$$(f \circ g)(x)(x) = x = (g \circ f)(x).$$

6. $f(x)$ and $f^{-1}(x)$ are symmetrical to $y = x$ since $D_f = R_{f^{-1}}$, $R_f = D_{f^{-1}}$. That is, if $f(x)$ passes through (a,b), $f^{-1}(x)$ passes through (b,a).

2.2 Quadratic Functions

1. The Standard Form:

$$y = ax^2 + bx + c,$$

where a is the coefficient of x^2, b is the coefficient of x, and c is the constant or y-intercept. $a, b, c \neq 0$.

- Zeros of the function (x-intercepts):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where $\Delta = b^2 - 4ac$ is the discriminant of the function.

- Equation of the line of symmetry & x-coordinate of the vertex

$$x = -\frac{b}{2a}.$$

- Vieta's Formula:

Theorem 2.2.1 Assume $x_1,\ x_2$ are two roots for equation $ax^2 + bx + c = 0$ $(a \neq 0)$, then

$$x_1 + x_2 = -\frac{b}{a};$$

$$x_1 \cdot x_2 = \frac{c}{a}.$$

- When $a > 0$, the parabola opens upwards.
  When $a < 0$, the parabola opens downwards.

2. Completion of square:

$$x^2 + px + \left(\frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2 = \left(x + \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2.$$

3. The Vertex Form:

$$y = a(x-h)^2 + k, \text{ where } (h,k) \text{ is the vertex.}$$

Example 2.2.1 Given that $f(x) = ax^2 + bx + c$, find the axis of symmetry and vertex.

$$
\begin{aligned}
f(x) &= a\left(x^2 + \frac{b}{a}x\right) + c \\
&= a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right] + c - \frac{b^2}{4a} \\
&= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}.
\end{aligned}
$$

$$
\begin{aligned}
\therefore \text{ axis of symmetry: } & x = -\frac{b}{2a} \\
\text{vertex: } & \left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right).
\end{aligned}
$$

2.3 Higher Order Polynomial Functions

1. Factor Theorem:

   Theorem 2.3.1 If $(x - a)$ is a factor of a polynomial $P(x)$, then $x = a$ must be a root for $P(x) \Rightarrow P(a) = 0$.

   Proof 2.3.1 Assume the quotient when $P(x)$ is divided by $(x - a)$ is $Q(x)$, then $P(x) = Q(x) \cdot (x - a)$. Then, $P(a) = Q(a) \cdot (a - a) = 0$.

2. Long division: solving polynomial equation.

   Example 2.3.1 For a cubic function, $P(x) = 2x^3 + bx^2 + cx + d,\ P(1) = P(2) = P(3) = 2$. **What is** $P(0)$**?**
   Since $P(1) = P(2) = P(3) = 2, Q(1) = Q(2) = Q(3) = 0,$ where $Q(x) = P(x) - 2$.
   Thus, $Q(x) = 2(x-1)(x-2)(x-3)$.

   $$\therefore P(x) = Q(2) + 2 = 2(x-1)(x-2)(x-3) + 2.$$

   $$\therefore P(0) = 2(-1)(-2)(-3) + 2 = -10.$$

3. Remainder Theorem:

   Theorem 2.3.2 When a polynomial $P(x)$ is divided by $(ax - b)$, the remainder R of this division must be

   $$P\left(\frac{b}{a}\right).$$

   Proof 2.3.2 Assume the quotient is $Q(x)$, and the reminder is R:

   $$P(x) = (ax - b)Q(x) + R.$$

   $$P\left(\frac{b}{a}\right) = 0 \cdot Q(X) + R = R.$$

4. Roots of Cubic Functions:

Theorem 2.3.3 For a cubic function $f(x) = ax^3 + bx^2 + cx + d$, given the roots of it are α, β, and γ. Then,

$$\begin{cases} \alpha+\beta+\gamma = -\frac{b}{a} \Rightarrow \sum\alpha = -\frac{b}{a} \\ \alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a} \Rightarrow \sum\alpha\beta = \frac{c}{a} \\ \alpha\beta\gamma = -\frac{d}{a} \Rightarrow \sum\alpha\beta\gamma = -\frac{d}{a} \end{cases}$$

Proof 2.3.3 Since α, β, γ are roots of $f(x)$,

$$f(x) = a(x-\alpha)(x-\beta)(x-\gamma).$$

$$\text{So } a(x-\alpha)(x-\beta)(x-\gamma) = ax^3 + bx^2 + cx + d,$$

$$\text{i.e., } ax^3 - a(\alpha+\beta+\gamma)x^2 + a(\alpha\beta+\alpha\gamma+\beta\gamma)x - a\alpha\beta\gamma = ax^3 + bx^2 + cx + d.$$

$$\Rightarrow \alpha+\beta+\gamma = -\frac{b}{a}, \alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}, \alpha\beta\gamma = -\frac{d}{a}.$$

Theorem 2.3.4

$$\sum\alpha = -\frac{b}{a}, \sum\alpha\beta = \frac{c}{a}, \sum\alpha\beta\gamma = -\frac{d}{a}, \sum\alpha\beta\gamma\delta = \frac{e}{a}.$$

2.4 Rational Functions

1. Reciprocal Functions: $f(x) = \frac{1}{x}$.

- Domain: $x \in \mathbb{R}$, $x \neq 0$
- As x increases, $\frac{1}{x}$ decreases $\Rightarrow x \to \infty, \frac{1}{x} \to 0$.
- Range: $y \in \mathbb{R}$, $y \neq 0$
- **Asymptotes**: $x = 0$, $y = 0$.
- Axis of symmetry: $y = x$, $y = -x$.
- **Self-inversing function**: have axis of symmetry $y = x$.

$$f(x) = f^{-1}(x).$$

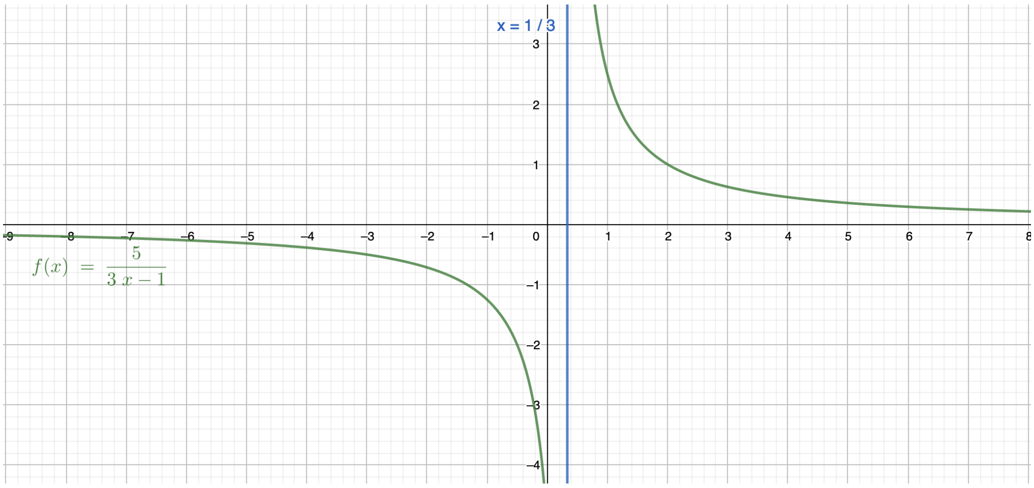
2. $y = \dfrac{a}{bx+c}$

- Vertical asymptotes (V.A.): $bx + c = 0$
- Horizontal asymptotes (H.A.): $y = 0$

Example 2.4.1 Draw the diagram of $y = \frac{5}{3x-1}$.
x-intercept: $0 = \frac{5}{3x-1} \Rightarrow$ no solution, no intercept.
H.A.: $y = 0$

y-intercept: $y = -5$

V.A.: $3x - 1 = 0,\ x = \dfrac{1}{3}$



3. $y = \dfrac{ax+b}{cx+d}$
   - V.A.: $cx + d = 0$
   - H.A.: $y = \dfrac{a}{c}$

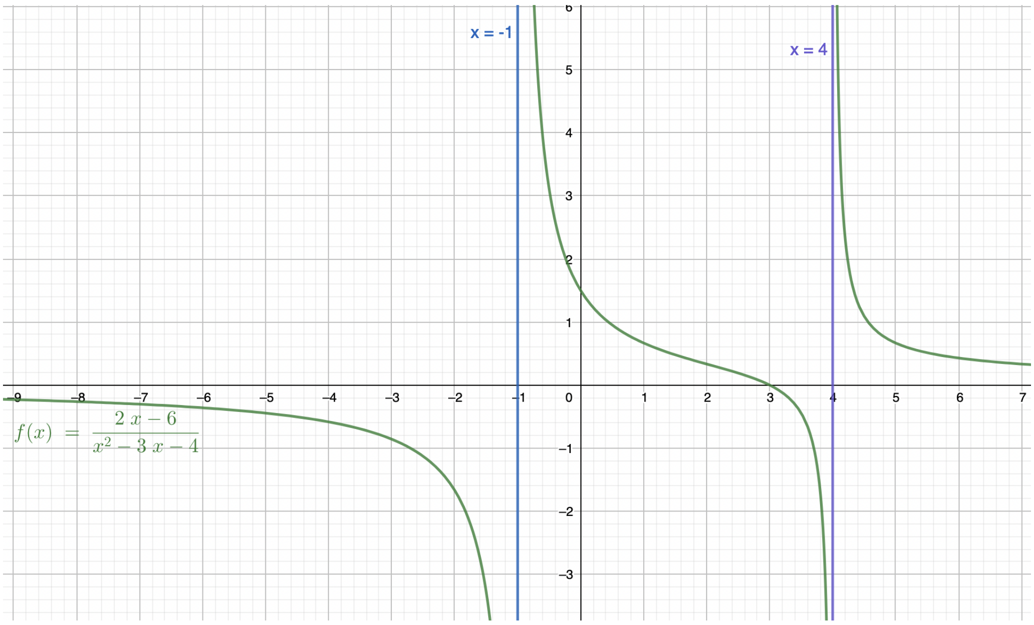
4. $y = \dfrac{ax+b}{cx^2+dx+e}$
   - V.A.: $cx^2 + dx + e = 0$
   - H.A.: As $x \to \pm\infty$, $\dfrac{ax}{cx^2} \to 0$, $y = 0$
   - Intercepts: $\left(0, \dfrac{e}{c}\right)$, $\left(-\dfrac{e}{d}, 0\right)$

Example 2.4.2 Draw the diagram of $y = \dfrac{2x-6}{x^2-3x-4}$**.**

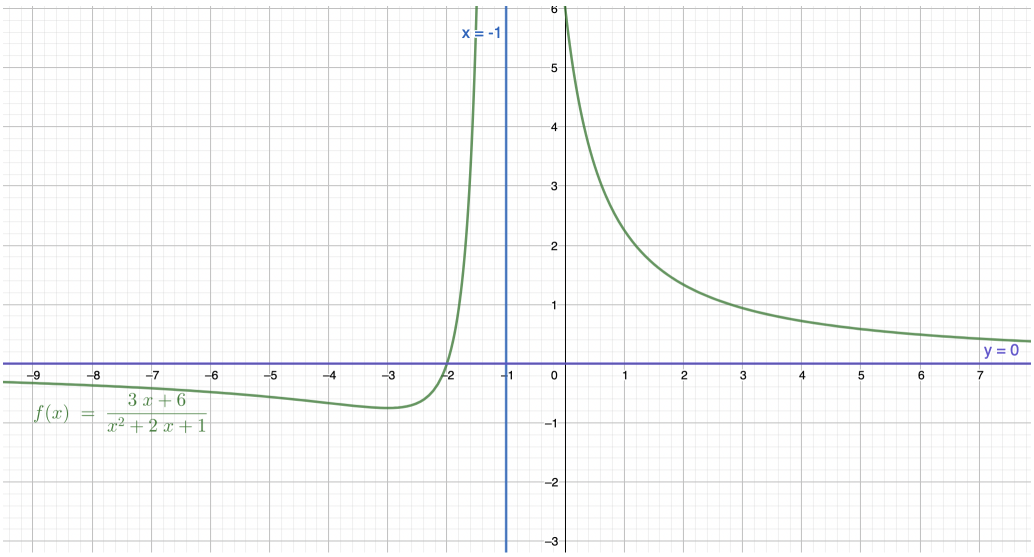
Intercept: $\left(0, \dfrac{3}{2}\right)$, $(3,0)$

H.A.: $y = 0$

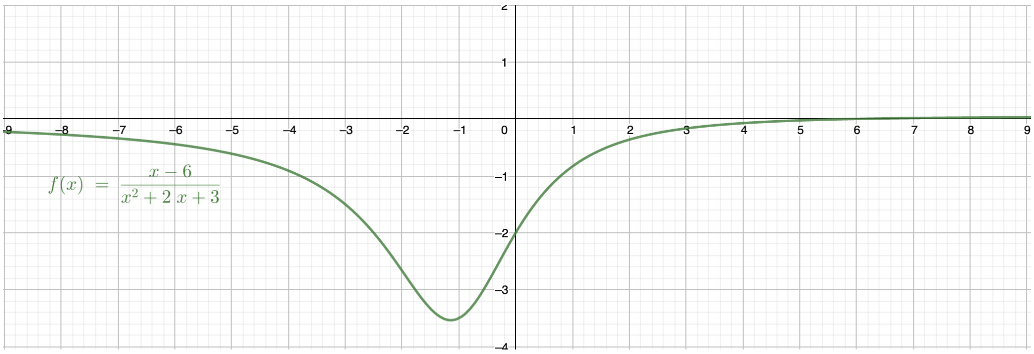
V.A.: $x = -1,\ x = 4$



Example 2.4.3 Draw the diagram of $y = \dfrac{3x+6}{x^2+2x+1}$**.**
Intercept: $(0,6)$, $(-2,0)$
H.A.: $y = 0$
V.A.: $x = -1$



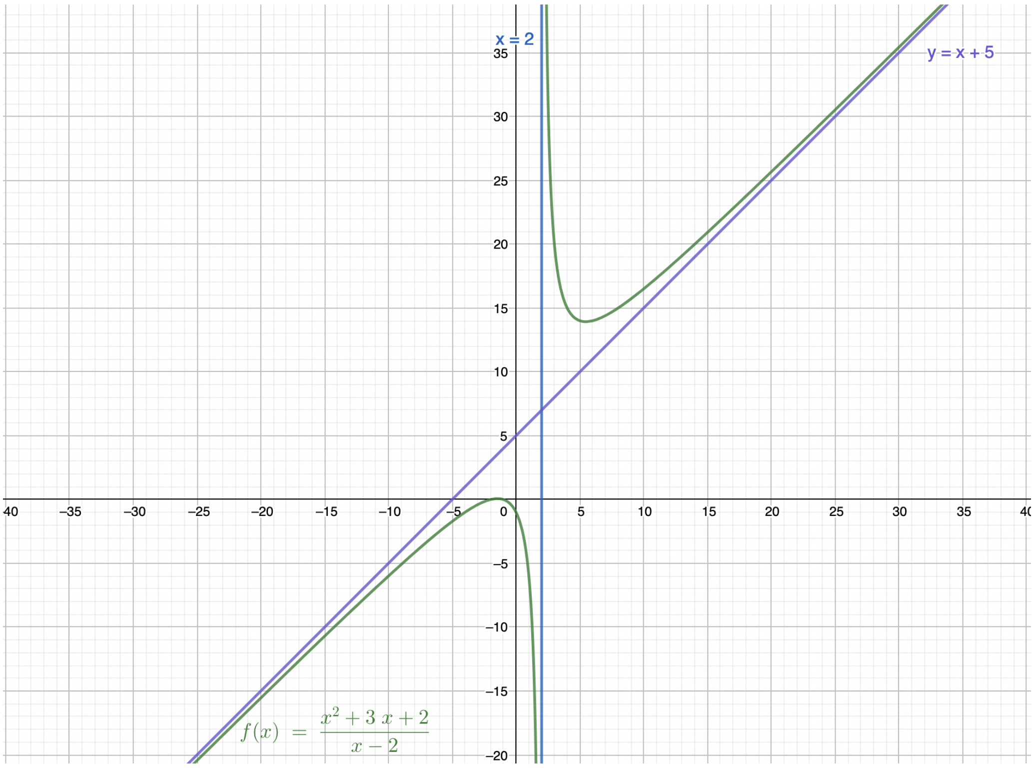
Example 2.4.4 Draw the diagram of $y = \dfrac{x-6}{x^2+2x+3}$**.**
Intercept: $(6,0)$, $(0,-2)$
When $x \to \infty$, $f(x)$ is positive. When $x \to -\infty$, $f(x)$ is negative.



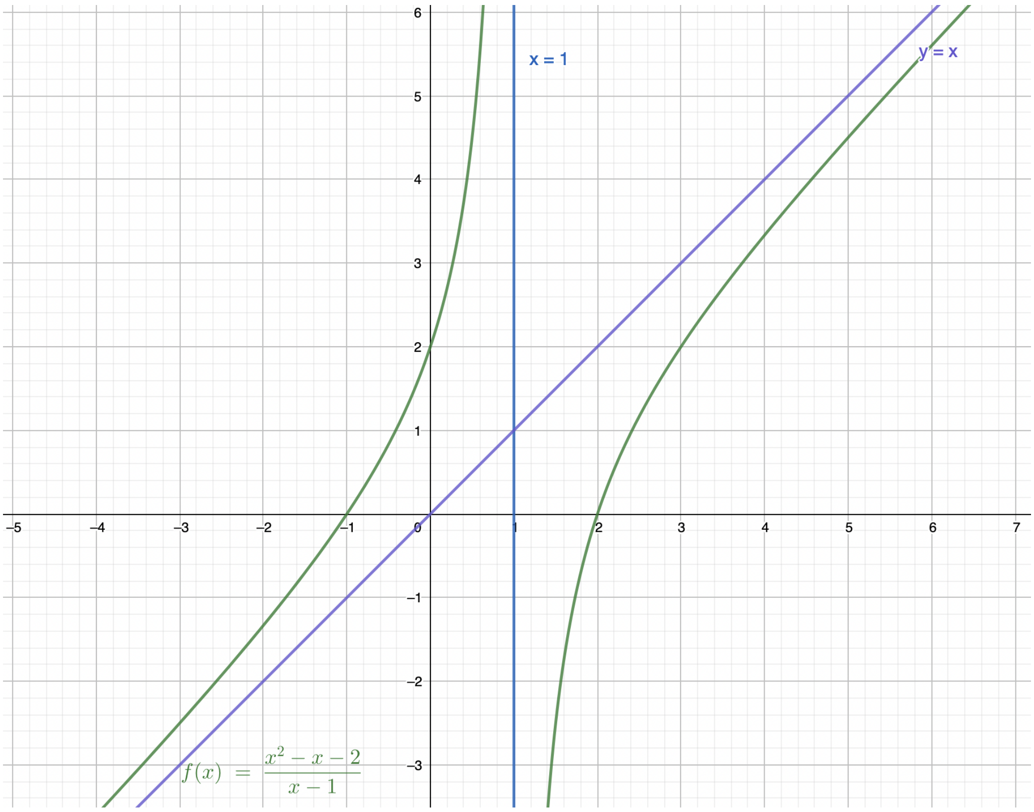
5. $y = \dfrac{ax^2+bx+c}{dx+e}$

- V.A.: $dx+e=0$
- **Oblique Asymptote**: Quotient of (ax^2+bx+c) divided by $(dx+e)$.
- Intercepts: $\left(0,\dfrac{c}{e}\right)$, $ax^2+bx+c=0$

Example 2.4.5 Draw the diagram of $y = \dfrac{x^2+3x+2}{x-2}$**.**
Intercept: $(0,-1)$, $(-1,0)$, $(-2,0)$
V.A.: $x=2$
O.A.: $y=x+5$ (Use long division)



Example 2.4.6 Draw the diagram of $y = \dfrac{x^2-x-2}{x-1}$**.**
Intercept: $(0,2)$, $(2,0)$, $(-1,0)$
V.A.: $x=1$
O.A.: $y=x$ (Use long division)



6. When the function has asymptotes:
   - Denominator$= 0$;
   - $\log_a 0$ (argument of a logarithm is 0)

2.5 Transformation of Functions

1. **Translation**:
   - $f(x+n)$ means translate $f(x)$ n units to the left.
   - $f(x-n)$ means translate $f(x)$ n units to the right.
   - $f(x)+n$ means translate $f(x)$ n units upwards.
   - $f(x)-n$ means translate $f(x)$ n units downwards.
2. Use translation vector to represent translation:
   A vector $\begin{pmatrix} a \\ b \end{pmatrix}$ means a units in the horizontal axis and b units in the vertical axis.

   Example 2.5.1 A translation vector $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ means $f(x+2)+3$, 2 units to the left and 3 units upwards.
3. **Reflections**:
   - $f(-x)$ reflects in the y-axis.
   - $-f(x)$ reflects in the x-axis.

- $f^{-1}(x)$ reflects in the $y = x$.
- $-f(-x)$ reflects in the origin.

4. **Stretches**:

   - $f(qx)$ is a horizontal stretch of a scale factor of $\frac{1}{q}$.
   - $pf(x)$ is a vertical stretch of a scale factor of p.

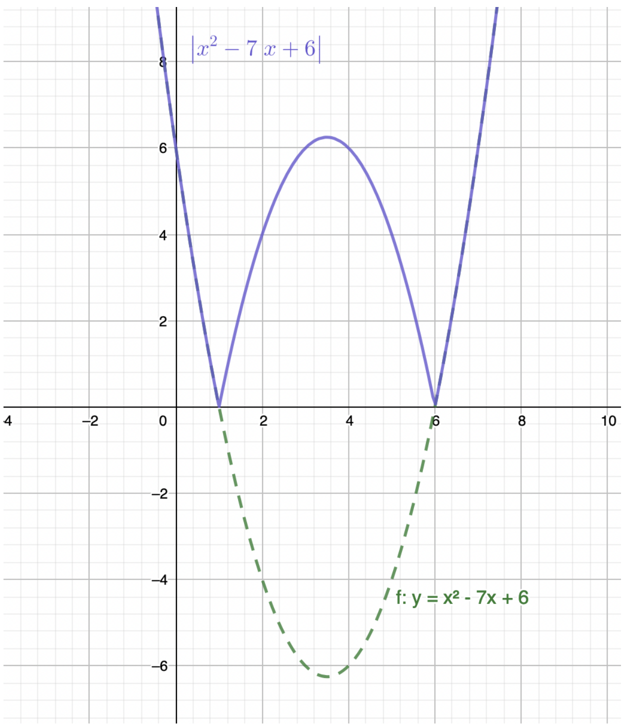
5. When a graph is transforming, the points shift but the connection remains.

6. Sequence of transformation:

   - Do the horizontal translation before the horizontal stretch.
   - The vertical translation is always after the vertical stretch.
   - Vertical stretch $\rightarrow$ Reflection $\rightarrow$ Horizontal translation $\rightarrow$ Horizontal stretch $\rightarrow$ Vertical translation

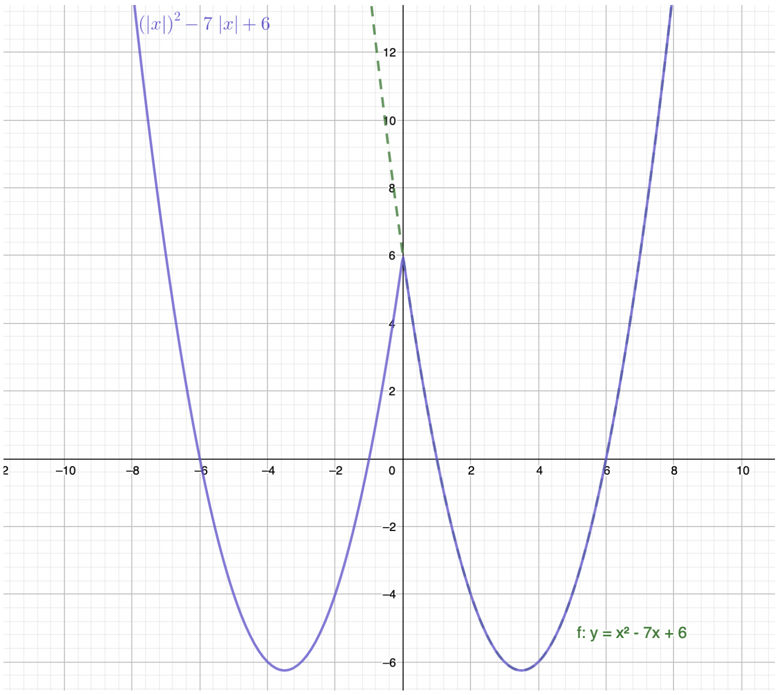
7. **Modulus Function**

   - $|f(x)|$: Fold everything below x-axis above x-axis.



Example 2.5.2

   - $f(|x|)$: Reflect everything on the right of y-axis to the left. Since $|x|$ must be positive, $|x| = |-x| \Rightarrow f(-x) = f(x)$, which is an even function.



Example 2.5.3

8. **Reciprocal of $f(x)$**

   - Table of Summary:

$f(x)$	$g(x) = \frac{1}{x}$
$f(a) = 0$	Line $x = a$ is vertical asymptote
Line $x = a$ is vertical asymptote	$g(a) = 0$
$f(x) \to \infty$	$g(x) \to 0$
$f(x) \to 0$	$g(x) \to \infty$
Line $y = b$ is horizontal asymptote	Line $y = \frac{1}{b}$ is horizontal asymptote
$f(x) = a$	$g(x) = \frac{1}{a}$

   - When $f(x)$ increases, $g(x)$ decreases.

2.6 Exponential and Logarithmic Functions

1. Exponential functions:

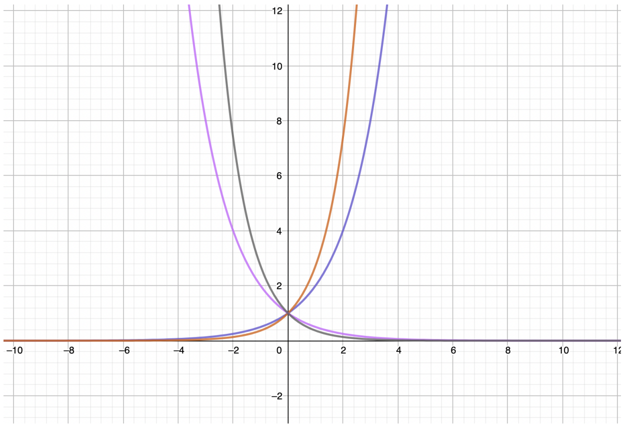
   - $f(x) = a^x$, $a > 1$ (increasing) and $0 < a < 1$ (decreasing).
   - $f(x) = a^x$ and $g(x) = \left(\frac{1}{a}\right)^x$ are symmetric to the y-axis.

   Proof 2.6.1

$$g(x) = \left(\frac{1}{a}\right)^x = (a^{-}1)^x = a^{-x} = f(-x).$$

   - Domain: $x \in \mathbb{R}$, Range: $y > 0$
   - Common point: $(0, 1)$; common H.A.: $y = 0$

- Graph:



2. Logarithmic functions:

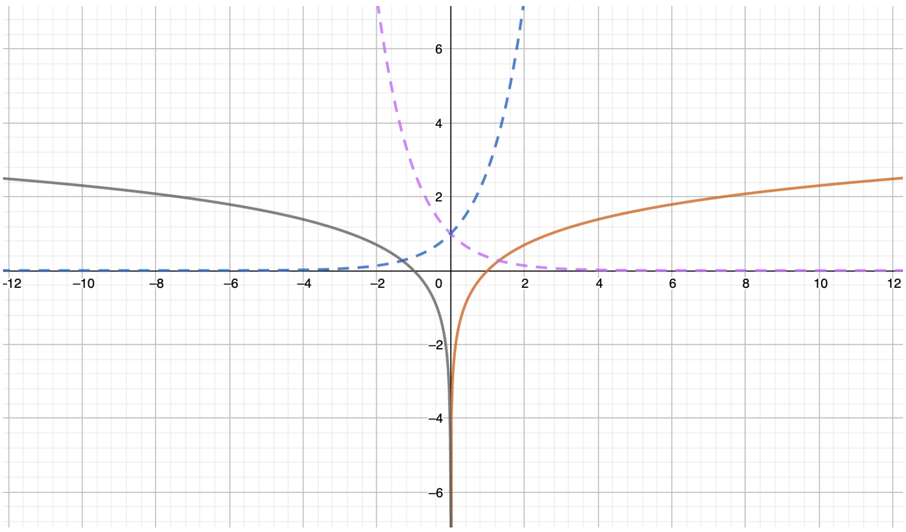
   - $f(x) = \log_a x = g^{-1}(x), g(x) = a^x$.
   - Common point: $(1,0)$; common V.A.: $x = 0$.
   - $f(x) = \log_a x$ and $g(x) = \log_{\frac{1}{a}} x$ are symmetric to the x-axis.

   Proof 2.6.2

$$\log_{\frac{1}{a}} x = \frac{\log_a x}{\log_{\frac{1}{a}} a} = \frac{\log_a x}{-1} = -\log_a x,$$

$$\therefore g(x) = \log_{\frac{1}{a}} x = -\log_a x = -f(x).$$

   - When $a > 1$, increasing function; when $0 < a < 1$, decreasing function.
   - Domain: $x > 0$, Range: $y \in \mathbb{R}$
   - Graph:



3. Solving logarithmic equations.

4. Solving exponential equations: take logarithm on both sides.

3 Trigonometry and Geometry

3.1 Trigonometry

3.1.1 Radian

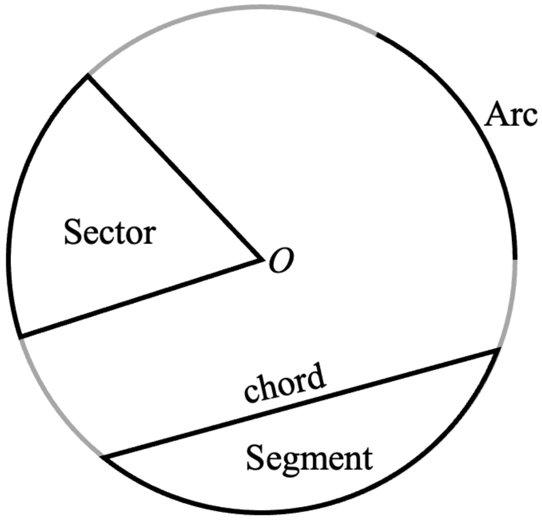
1. Radian as the unit of angle:

   - $$\pi \text{ rad} = 180^\circ$$
   - rad can be omitted. i.e., $\widehat{A} = 1$ means angle A is 1 radian.
   - Unit conversion:

   $$\text{degree} \times \frac{\pi}{180^\circ} = \text{radian; radian} \times \frac{180^\circ}{\pi} = \text{degree.}$$

2. Arc:

   - The **circumference** (perimeter) is $2\pi r$.



   - If the angle of the arc is θ (in radian), the length of arc$(l) = r \cdot \theta$.
   - The area of a sector:

   $$A = \frac{1}{2} r^2 \theta.$$

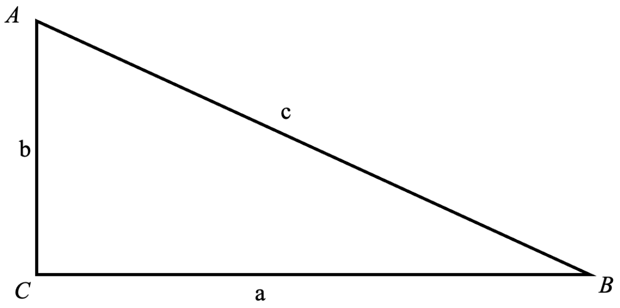
   - The area of a segment:

   $$A = \frac{1}{2} r^2 (\theta - \sin\theta).$$

   (Proof: the area of the triangle according to the sine rule is $\frac{1}{2} ab \sin C$)

3.1.2 Solution of Triangle

1. Define sine, cosine, and tangent:



Definition 3.1.1

$$\sin A = \frac{a}{c},\ \sin B = \frac{b}{c};$$
$$\cos A = \frac{b}{c},\ \cos B = \frac{a}{c};$$
$$\tan A = \frac{a}{b},\ \tan B = \frac{b}{a}.$$

2. The Sine Rule:

Theorem 3.1.1

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

- The bigger the angle, the longer the side.
- Area of a triangle:

$$A = \frac{1}{2}ab\sin C.$$

3. The Cosine Rule:

Theorem 3.1.2

$$b^2 + c^2 - a^2 = 2bc \cdot \cos A;$$
$$a^2 + c^2 - b^2 = 2ac \cdot \cos B;$$
$$a^2 + b^2 - c^2 = 2ab \cdot \cos C.$$

4. Inverse Trigonometric Functions:

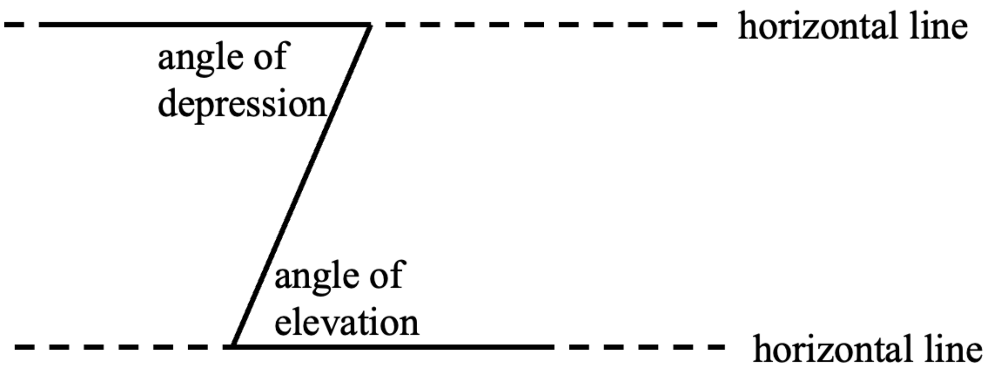
Definition 3.1.2

$$\sin^{-1}\theta = \arcsin\theta;$$
$$\cos^{-1}\theta = \arccos\theta;$$
$$\tan^{-1}\theta = \arctan\theta.$$

5. Ambiguity of Sine Rule:

$$\sin\theta = \sin(180^\circ - \theta) \text{ OR } \sin\theta = \sin(\pi - \theta).$$

6. Angle of Elevation and Depression:

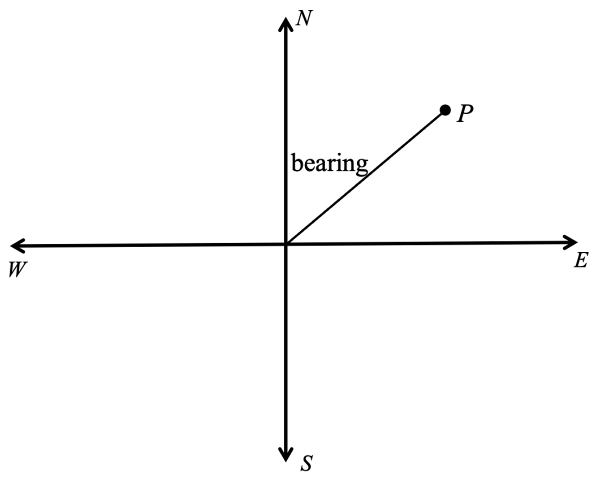


Definition 3.1.3 • **Angle of Elevation** is the angle "up" from horizontal.

- **Angle of Depression** is the angle "down" from horizontal.

7. Bearing:

   - Bearing is a way of describing direction.
   - All bearings are measured clockwise from the North direction.

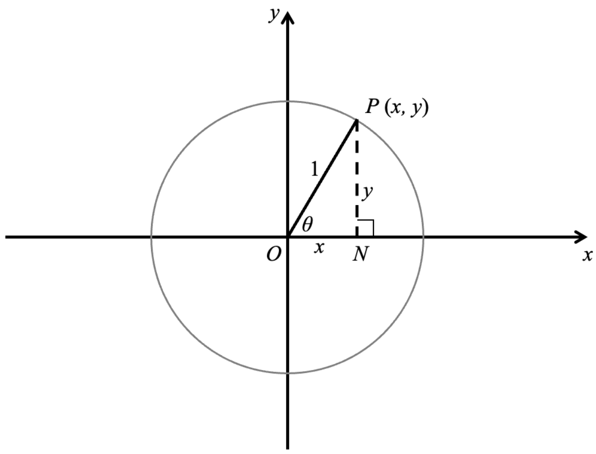


   - Bearing of A from B: construct at B.
     N.B.: Bearing of A from B is different from bearing of B from A.

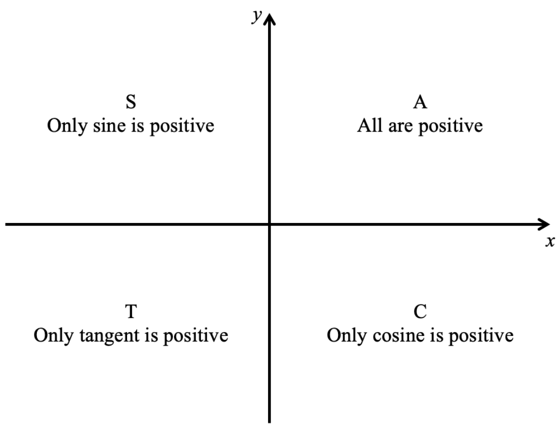
3.1.3 Definition of Trigonometric Function

1. Unit Circle:

   - Center at $(0,0)$ with a radius of 1.



- If an angel θ opens in a counterclockwise direction, then θ is positive.
  If an angle θ opens in a clockwise direction, then θ is negative.
- In the diagram, $\theta = \theta + 2k\pi,\ k \in \mathbb{Z}$.
- $$\sin\theta = \frac{PN}{OP} = \frac{y}{1} = y;$$
  $$\cos\theta = \frac{ON}{OP} = \frac{x}{1} = x;$$
  $$\tan\theta = \frac{PN}{ON} = \frac{y}{x} = \frac{\sin\theta}{\cos\theta};$$
- In Q_1 and Q_2, $\sin\theta$ will be positive.
  In Q_1 and Q_4, $\cos\theta$ will be positive.
  In Q_1 and Q_3, $\tan\theta$ will be positive.
  $\Rightarrow$ CAST:



2. Special Angles:

$$\sin 0^\circ = 0 = \cos 90^\circ$$
$$\sin 30^\circ = \frac{1}{2} = \cos 60^\circ$$
$$\sin 45^\circ = \frac{\sqrt{2}}{2} = \cos 45^\circ$$
$$\sin 60^\circ = \frac{\sqrt{3}}{2} = \cos 30^\circ$$
$$\sin 90^\circ = 1 = \cos 0^\circ$$

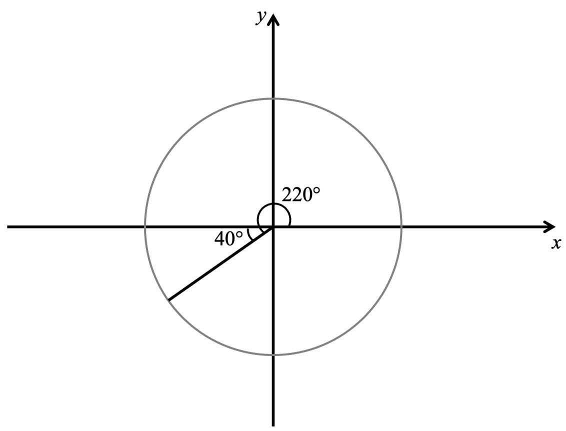
$$\tan 0^\circ = \frac{\sin 0^\circ}{\cos 0^\circ} = 0$$
$$\tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ} = \frac{\sqrt{3}}{3}$$
$$\tan 45^\circ = \frac{\sin 45^\circ}{\cos 45^\circ} = 1$$
$$\tan 60^\circ = \frac{\sin 60^\circ}{\cos 60^\circ} = \sqrt{3}$$
$$\tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ} = \infty$$

3. Relative Acute Angles (RAA):
   - Acute angle is the angle with x-axis.
   - The absolute value of angles have the same acute angle is the same.

Example 3.1.1 (a) $30^\circ,\ 150^\circ,\ 210^\circ,\ 330^\circ$ have the same acute angle.
$\therefore |\sin 30^\circ| = |\sin 150^\circ| = |\sin 210^\circ| = |\sin 330^\circ|$.

(b)

$$\tan 220^\circ = \tan 40^\circ;\ \cos 215^\circ = -\cos 35^\circ$$



3.1.4 Trigonometric Identity

1. Pythagorean's Identity:

$$\sin^2\theta + \cos^2\theta \equiv 1.$$

Proof 3.1.1

$$a^2 + b^2 = c^2 \Rightarrow \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$$
$$\Rightarrow \sin^2\theta + \cos^2\theta = 1.$$

2. Definition of Tangent:

- $$\tan\theta = \frac{\sin\theta}{\cos\theta};$$
- $$\cot\theta = \frac{1}{\tan\theta};$$
- $$\sec\theta = \frac{1}{\cos\theta};$$
- $$\csc\theta = \frac{1}{\sin\theta}.$$

3. Extended Pythagorean's Identity:

$$\tan^2\theta + 1 = \sec^2\theta;$$

$$\cot^2\theta + 1 = \csc^2\theta.$$

Proof 3.1.2

$$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \Rightarrow \tan^2 + 1 = \sec^2\theta;$$
$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \Rightarrow \cot^2\theta + 1 = csc^2\theta.$$

N.B.: a reflex angle is an angle bigger than 180°, smaller than 360°.

4. Compound Angle Formula:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B;$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B;$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B;$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B;$$

Example 3.1.2 Find the exact value of $\cos\frac{\pi}{12}$**.**

$$\begin{aligned}\cos\frac{\pi}{12} &= \cos\frac{\pi}{4} - \frac{\pi}{6}\\ &= \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}\\ &= \frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\times\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}.\end{aligned}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}.$$

Proof 3.1.3

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$
$$=\frac{\frac{\sin A\cos B}{\cos A\cos B}+\frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B}-\frac{\sin A\sin B}{\cos A\cos B}}$$
$$=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}.$$

5. In the linear function $y=mx+b$, $m=\tan\theta$, where θ is the angle between the line and the positive x-axis.

6. Double Angle Formula:

$$\cos(2\theta)=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1=1-2\sin^2\theta;$$

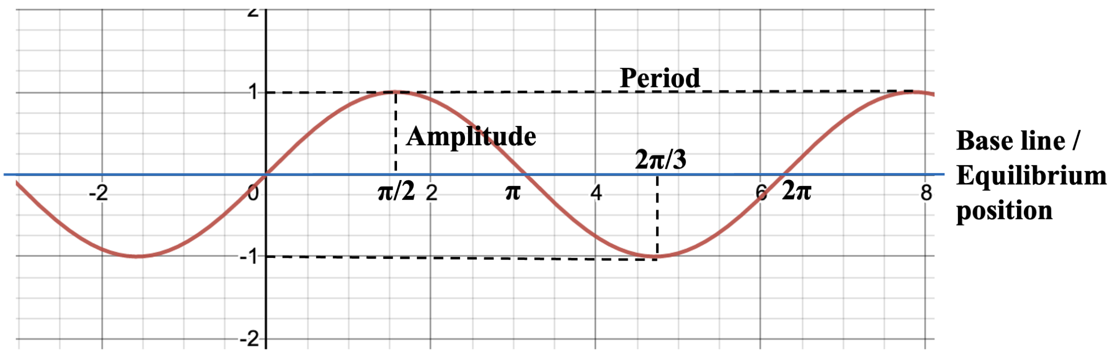
$$\sin(2\theta)=2\sin\theta\cos\theta;$$

$$\tan(2\theta)=\frac{2\tan\theta}{1-\tan^2\theta}.$$

7. Proving Identities.

3.1.5 Trigonometric Functions and Transformation

1. Sine: Odd function: $\sin(-x)=-\sin x$.



$$\text{T(Period)}=2\pi;$$

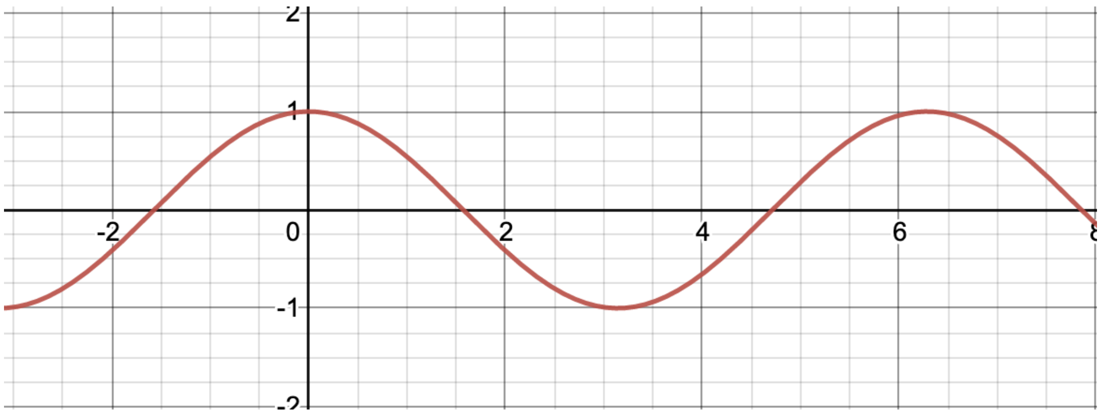
$$\text{Base line}=0;$$

$$\text{Amplitude}=\left|\frac{y_{\max}-y_{\min}}{2}\right|=1;$$

$$\text{Range: }\sin x\in[-1,\ 1];$$

$$\text{Domain: }x\in\mathbb{R}.$$

2. Cosine: Even function: $\cos(-x) = \cos x$.



$$\text{T(Period)} = 2\pi;$$

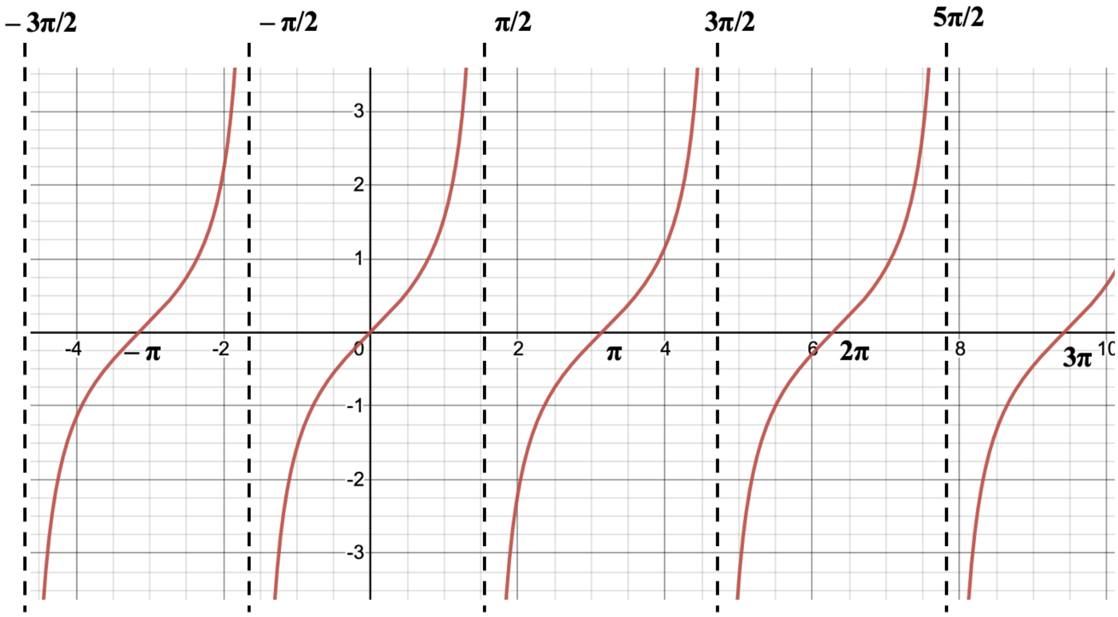
$$\text{Base line} = 0;$$

$$\text{Amplitude} = \left| \frac{y_{\max} - y_{\min}}{2} \right| = 1;$$

$$\text{Range: } \cos x \in [-1,\ 1];$$

$$\text{Domain: } x \in \mathbb{R}.$$

3. Tangent:



$$\text{T(Period)} = \pi;$$

$$\text{No amplitude(A);}$$

$$\text{V.A.: } x = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z};$$

$$\text{Range: } \tan x \in \mathbb{R};$$

$$\text{Domain: } x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}.$$

4. Transformation of Sine and Cosine:

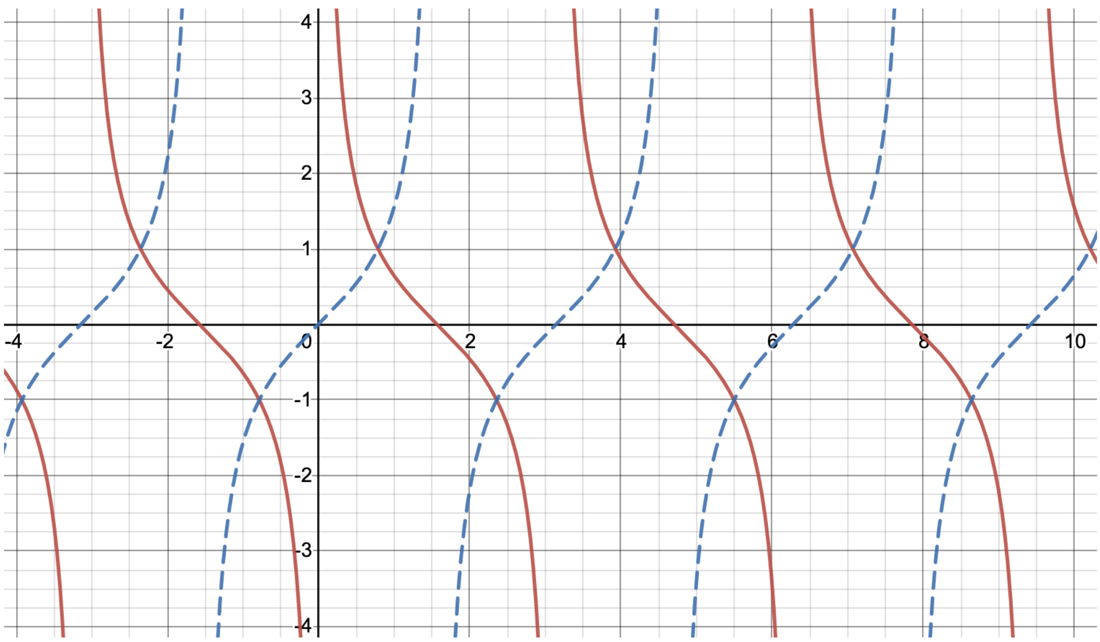
$$y = A\sin(\omega(x-\varphi)) + h.$$

- Horizontal stretch with the scale factor of $\frac{1}{\omega}$. $\Rightarrow$ changes $T = \frac{2\pi}{\omega}$.
- Horizontal translate to the right φ units. $\Rightarrow$ changes the initial point to $(\varphi, 0)$.
- Vertical stretch with a scale factor of A. $\Rightarrow$ changes the amplitude= $|A|$.
- Vertical translation of h units upwards. $\Rightarrow$ changes the equilibrium position $y = h$.
- Range of $y = A\sin(\omega(x-\varphi)) + h$: $y \in [h-A, h+A]$.

$$y = A\cos(\omega(x-\varphi)) + h.$$

- Horizontal stretch with the scale factor of $\frac{1}{\omega}$. $\Rightarrow$ changes $T = \frac{2\pi}{\omega}$.
- Horizontal translate to the right φ units. $\Rightarrow$ changes the initial point to $(\varphi, 1)$.
- Vertical stretch with a scale factor of A. $\Rightarrow$ changes the amplitude= $|A|$, initial point (φ, A).
- Vertical translation of h units upwards. $\Rightarrow$ changes the equilibrium position $y = h$, initial point $(\varphi, A+h)$.

5. Cotangent:

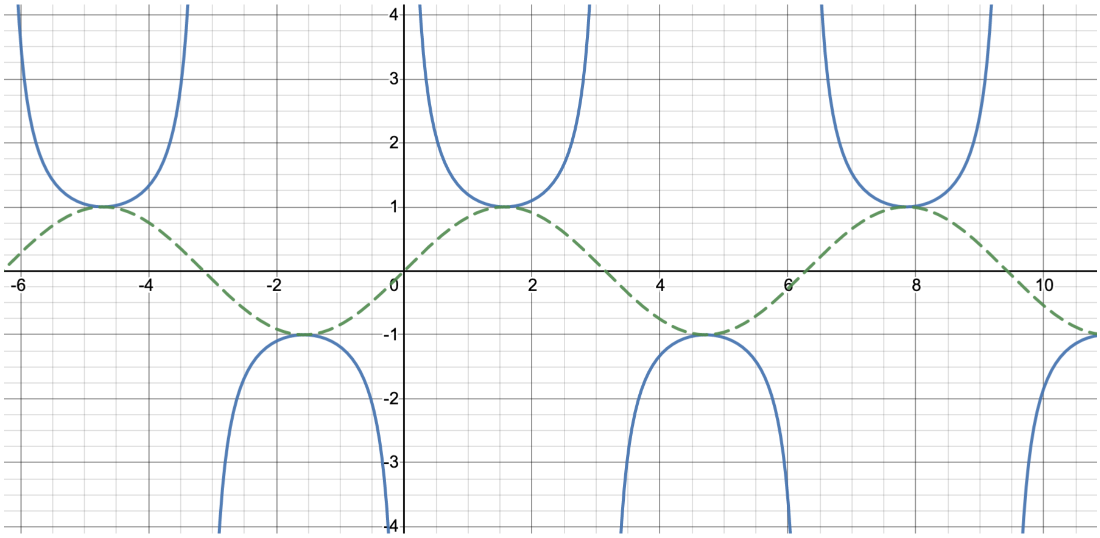


$$\text{V.A.: } x = k\pi$$

$$\text{Period: } \pi$$

$$\text{Pass through } \left(\frac{\pi}{2} + k\pi, 0\right)$$

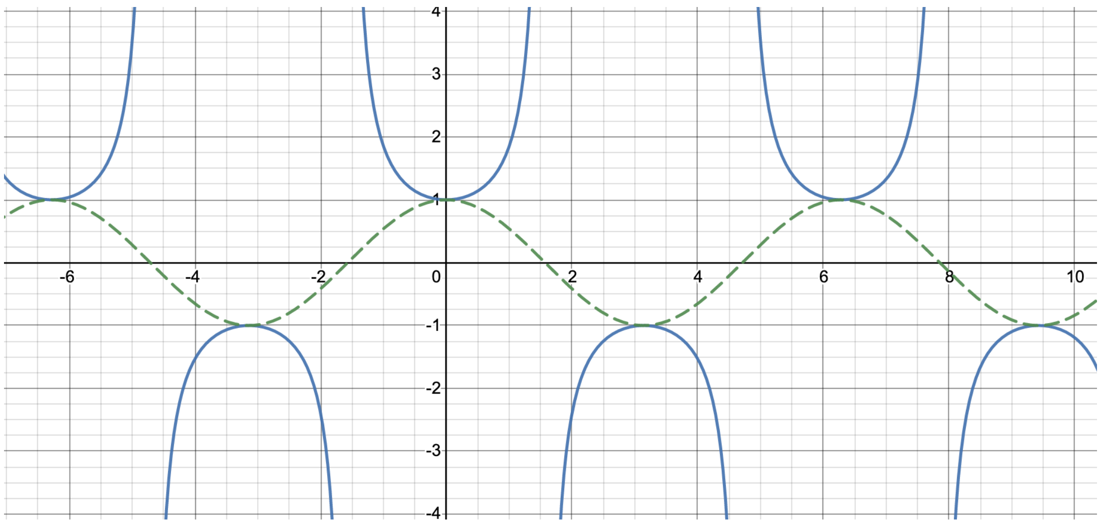
6. Cosecant:



$$\text{Domain: } x \neq k\pi$$

$$\text{Range: } y \in \,]-\infty, -1\,[\,\cup\,]\,1, +\infty\,[$$

7. Secant:



$$\text{Domain: } x \neq \frac{\pi}{2} + k\pi$$

$$\text{Range: } y \in \,]-\infty, -1\,[\,\cup\,]\,1, +\infty\,[$$

8. When drawing the graph of *secx* and $\csc x$, draw *cosx* and $\sin x$ first.

9. Conversion between sine and cosine:

- $$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

- $$\cos\left(\frac{\pi}{2}-x\right)=\sin x$$

- $$\cos\left(\frac{\pi}{2}+x\right)=\cos\left[\pi-\left(\frac{\pi}{2}-x\right)\right]=-\cos\left(\frac{\pi}{2}-x\right)=-\sin x$$

- $$\sin\left(\frac{\pi}{2}+x\right)=\sin\left[\pi-\left(\frac{\pi}{2}-x\right)\right]=\sin\left(\frac{\pi}{2}-x\right)=\cos x$$

3.1.6 Solving Trigonometric Functions

1. Solving Trigonometric Functions in Paper 1:

   - Values of special angles
   - From relative acute angles and CAST rule
   - Modification of period
   - Check the solution with domain

   Example 3.1.3 Solve for $\cos x=\frac{\sqrt{3}}{2}$ for $0<x<3\pi$.

   $$\text{Consider } x\in[0,2\pi]$$
   $$x=\frac{\pi}{6},\frac{11\pi}{6}.$$
   $$\text{In the domain of } x\in[0,3\pi],$$
   $$\text{Another solution is } \frac{13\pi}{6}.$$

2. Transformed Trigonometric Equations:

   Example 3.1.4 Solve $6\sin\left(2\left(x-\frac{\pi}{6}\right)\right)-2=1,\ \frac{\pi}{6}<x<2\pi$.

   $$\sin\left(2\left(x-\frac{\pi}{6}\right)\right)=\frac{1}{2}.$$
   $$\text{Let } t=2\left(x-\frac{\pi}{6}\right):$$
   $$\because\frac{\pi}{6}<x<2\pi,$$
   $$\therefore 0<2\left(x-\frac{\pi}{6}\right)<\frac{11\pi}{3},\ 0<t<\frac{11\pi}{3}.$$
   $$\sin t=\frac{1}{2}\ \Rightarrow\ t=\frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{13\pi}{6},\ \frac{17\pi}{6};$$
   $$\Rightarrow x=\frac{\pi}{4},\ \frac{7\pi}{12},\ \frac{5\pi}{4},\ \frac{19\pi}{12}.$$

3. Solving Trigonometric Functions in Paper 2:

   - Change mode to RADIAN.

- Plot the functions.
- Adjust the window.
- Calculate the intersects.
- Repeat step 4 if necessary.

3.1.7 Inverse Trigonometric Functions

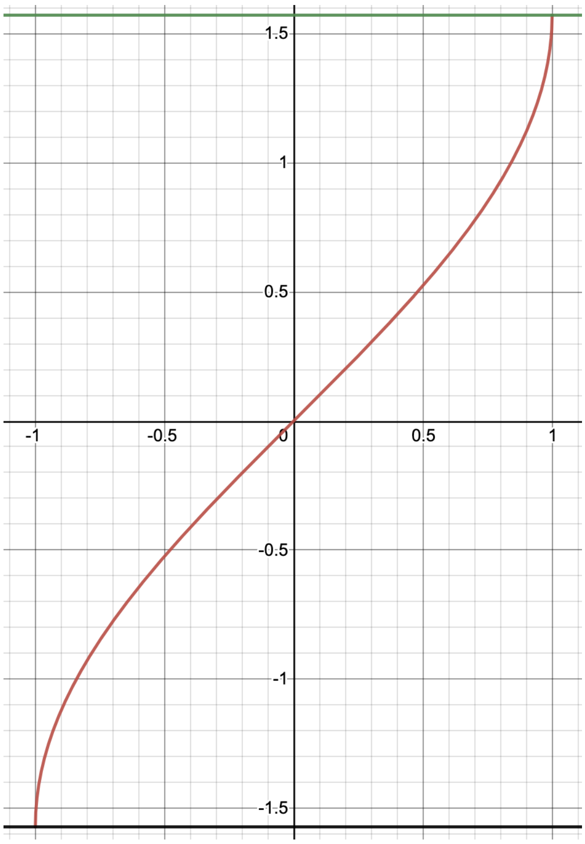
1. Inverse Trigonometric Function:

   - $$y = \arcsin x$$
   - $$y = \arccos x$$
   - $$y = \arctan x$$
   - $$\operatorname{arcsec} x = \arccos\left(\frac{1}{x}\right)$$
   - $$\operatorname{arccsc} x = \arcsin\left(\frac{1}{x}\right)$$
   - $$\operatorname{arccot} x = \arctan\left(\frac{1}{x}\right)$$

2. One-to-one Function:

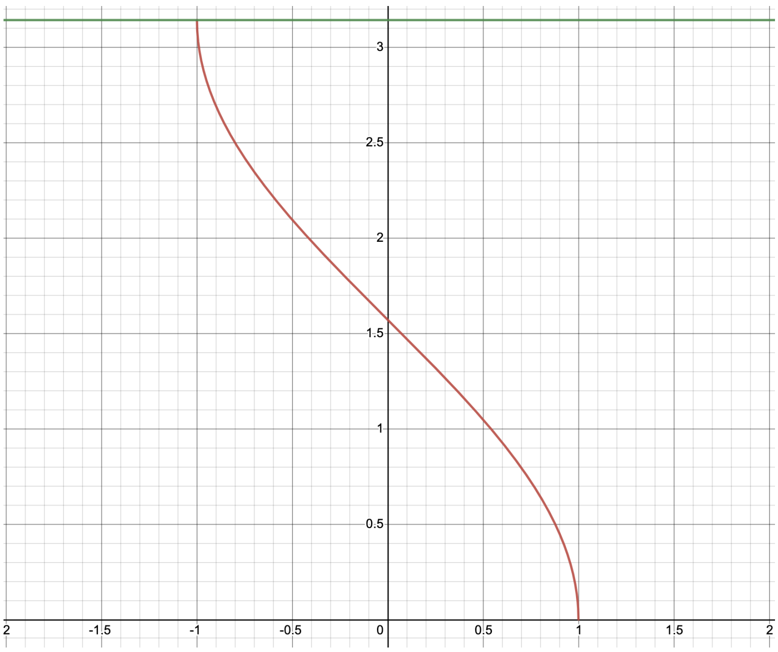
   - In order for functions to have the inverse function, it must be so-called **one-to-one** function (bijection).
   - One x value to one (and only one) y value.
     One y value to one (and only one) x value.

3. Domain and range for $\arcsin x$:



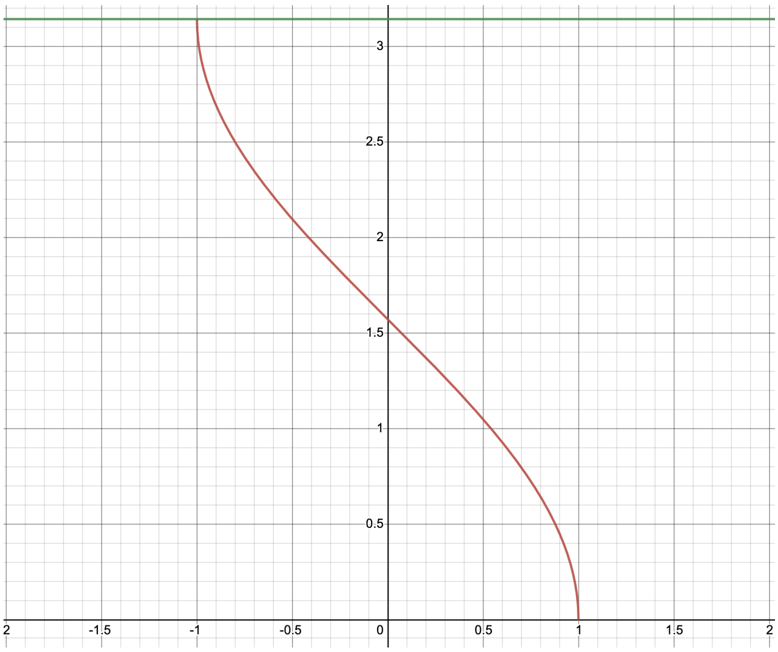
- Domain: $x \in [-1, 1]$ (Range $\sin x \in [-1, 1]$).
- Range: $\arcsin x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (Domain $\sin x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$).

4. Domain and range for $\arccos x$:



- Domain: $x \in [-1, 1]$.
- Range: $\arccos x \in [0, \pi]$.

5. Domain and range for $\arctan x$:



- Domain: $x \in \mathbb{R}$
- Range: $y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

3.2 Vectors

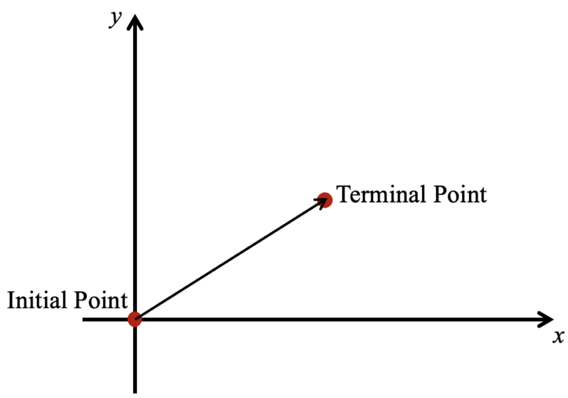
3.2.1 Introduction to Vectors

1. Vector:

   Definition 3.2.1 A **vector** is a quantity with a direction and magnitude. It is noted as $\vec{a}$.

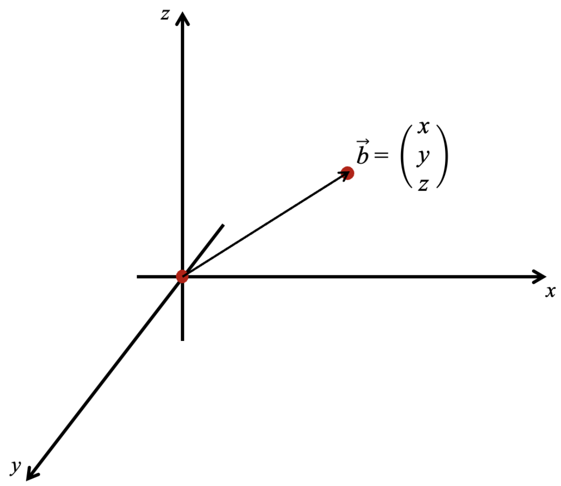
2. Components of a vector:

   - 2-D:



Example 3.2.1 The vector $\vec{a} = \binom{3}{2}$ means 3 units in the horizontal direction and 2 units in the vertical direction.

   - 3D:



3. **Magnitude/Modulus** of vector:

   - 2D:

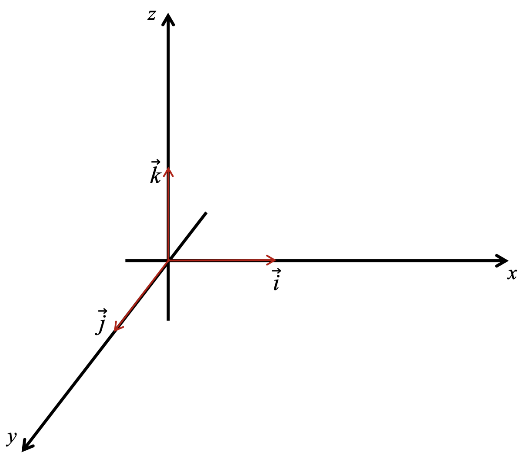
$$\text{For } \vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}, \ |\vec{a}| = \sqrt{x^2 + y^2}.$$

   - 3D:

$$\text{For } \vec{b} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \ \left|\vec{b}\right| = \sqrt{x^2 + y^2 + z^2}.$$

4. **Unit Vector**: A vector of length 1:

   - $\vec{i}$: unit vector on the x-axis.
   - $\vec{j}$: unit vector on the y-axis.
   - $\vec{k}$: unit vector on the z-axis.



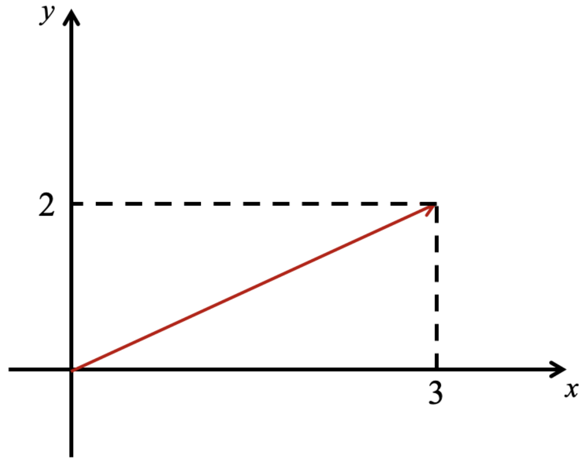
5. Sum of vectors:

   - **Position vector**: A vector that has an initial point at the origin.

Example 3.2.2

$$\vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\vec{a} = 3\vec{i} + 2\vec{j}.$$



- Let $\vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} m \\ n \end{pmatrix}$

$$\vec{a} + \vec{b} = \begin{pmatrix} x+m \\ y+n \end{pmatrix}.$$

6. Multiplication of vectors by a scalar:

   Let $\vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ and n be a scalar:

$$n\vec{a} = n\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} nx \\ ny \end{pmatrix}.$$

   $n\vec{a}$ and $\vec{a}$ are in the same direction $\Rightarrow$ parallel.

7. Subtracting a vector:

   Let $\vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}, \vec{b} = \begin{pmatrix} m \\ n \end{pmatrix}$.

$$\vec{a} - \vec{b} = \begin{pmatrix} x-m \\ y-n \end{pmatrix}.$$

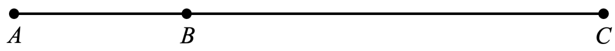
Proof 3.2.1

$$-\vec{b} = (-1)\vec{b} = \begin{pmatrix} -m \\ -n \end{pmatrix}$$

$$\vec{a} - \vec{b} = \vec{a} + \left(-\vec{b}\right) = \begin{pmatrix} x-m \\ y-n \end{pmatrix}.$$

8. **Zero vector**: $\vec{0}$.

9. **Collinear points**: three points, A, B, and C, are said to be collinear if $\vec{AB} = t\vec{AC}$.

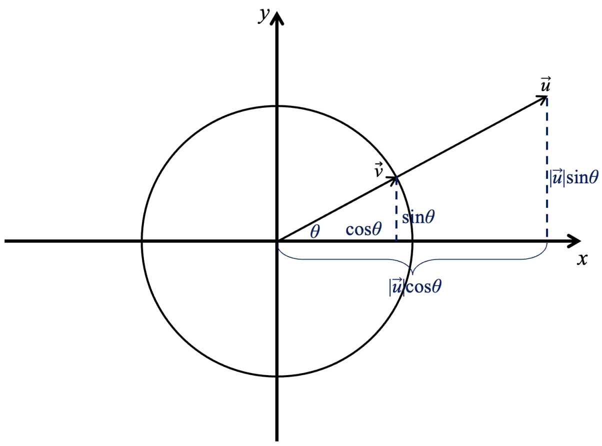


10. Find a unit vector parallel to $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$.

    - Find the value $|\vec{u}|$.
    - Then, the unit vector parallel to $\vec{u}$ is

$$\vec{v} = \frac{\vec{u}}{|\vec{u}|}.$$

11. Vectors and unit circle:



θ is the angle with the horizontal axis. The unit vector $\vec{v}$, in the same direction as $\vec{u}$ is:

$$\vec{v} = \cos\theta \cdot \vec{i} + \sin\theta \cdot \vec{j}$$

$$\vec{v} = \frac{1}{|\vec{u}|} \cdot \vec{u} \Rightarrow \vec{u} = |\vec{u}| \cdot \vec{v} = |\vec{u}| \cos\theta \cdot \vec{i} + |\vec{u}| \sin\theta \cdot \vec{j}$$
$$= |\vec{u}| \left(\cos\theta \cdot \vec{i} + \sin\theta \cdot \vec{j}\right).$$

3.2.2 Scalar Product and Its Properties

1. The **scalar product** of two vectors is a real number (scalar).

    - The algebraic definition:

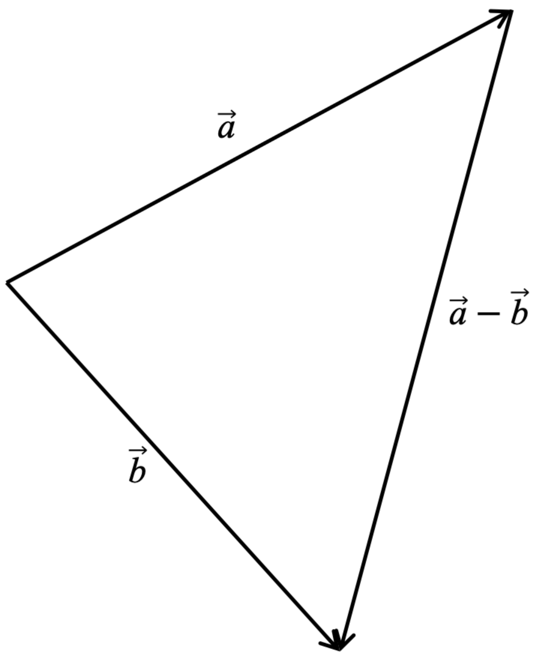
      For $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$,

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2.$$

The scalar product is also called the dot product.

- The geometric definition:
  For $\vec{a}$ and $\vec{b}$,

$$\vec{a}\cdot\vec{b}=|\vec{a}|\left|\vec{b}\right|\cos\theta,\ \theta \text{is the angle between the two vectors.}$$



Proof 3.2.2 By cosine rule:

$$\begin{aligned}\left|\vec{b}-\vec{a}\right|^2 &= |\vec{a}|^2+\left|\vec{b}\right|^2-2|\vec{a}|\left|\vec{b}\right|\cos\theta\\ \left|\vec{b}\right|^2-2\vec{a}\vec{b}+|\vec{a}|^2 &= |\vec{a}|^2+\left|\vec{b}\right|^2-2|\vec{a}|\left|\vec{b}\right|\cos\theta\\ \therefore \vec{a}\cdot\vec{b} &= |\vec{a}|\left|\vec{b}\right|\cos\theta.\end{aligned}$$

- Combining the two definitions:

$$\cos\theta=\frac{a_1b_1+a_2b_2}{\sqrt{\left(a_1^2+a_2^2\right)\left(b_1^2+b_2^2\right)}}.$$

2. 3-D vectors: $\vec{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\vec{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$:

- $$\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3.$$

- $$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{\left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right)}}.$$

3. Properties of scalar product:

   - If $\vec{a}\cdot\vec{b} = 0 \;\Rightarrow\; \begin{cases} \vec{a} = 0 \\ \vec{b} = 0 \\ \vec{a} \text{ and } \vec{b} \text{ are perpendicular (orthogonal)} \;\Rightarrow\; \theta = \frac{\pi}{2} \end{cases}.$

   - If $\vec{a}$ and $\vec{b}$ are collinear,
   $$\vec{a}\cdot\vec{b} = \pm|\vec{a}|\left|\vec{b}\right|.$$

   Proof 3.2.3 Angel between $\vec{a}$ and $\vec{b}$ is $0°$.
   $\cos 0° = 1 \;\Rightarrow\; \vec{a}\cdot\vec{b} = |\vec{a}|\left|\vec{b}\right|$ for $\vec{a}$, $\vec{b}$ at the same direction.
   OR $\vec{a}\cdot\vec{b} = -|\vec{a}|\left|\vec{b}\right|$ for $\vec{a}$ and $\vec{b}$ at opposite directions.

   - $$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}.$$

   - $$\vec{a}\cdot\vec{a} = |\vec{a}|^2$$

   Proof 3.2.4
   $$\vec{a}\cdot\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = a_1^2 + a_2^2 = |\vec{a}|^2.$$

   - $$\vec{a}\cdot\left(\vec{b} + \vec{c}\right) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}.$$

   - $$\lambda\left(\vec{a}\cdot\vec{b}\right) = (\lambda\vec{a})\cdot\vec{b} = \vec{a}\cdot\left(\lambda\vec{b}\right).$$

3.2.3 Vector Equation of a Line

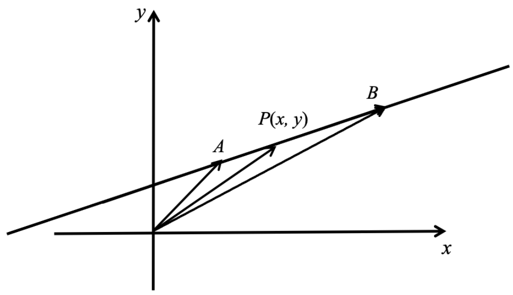
1. There is only one line that passes through two distinct points.

   Theorem 3.2.1 In the coordinate plane, the equation can be found as:
   For $A(x_1, y_1)$ and $B(x_2, y_2)$, the line passes through A, B is given by
   $$y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1.$$

2. **Slope, y-intercept form**: $y = mx + k$, where m is the slope, and k is the y-intercept.
   It can be rearranged to $ax + by = c$; $a, b, c \in \mathbb{R}$, where a and b cannot be equal to 0 at the same time.

3. **Vector form** of a line:

- For every point $P(x,y)$ that lies on the line AB, the vector $\overrightarrow{AP}$ must be collinear or parallel to $\overrightarrow{AB}$: $\overrightarrow{AP} = k\overrightarrow{AB},\ k \in \mathbb{R}$.



  (a) The vector $\overrightarrow{AB}$ is called a **direction vector** of the line.
  All the vectors that are parallel to $\overrightarrow{AB}$ can also define the same line.

  (b) Assume $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OP} = \vec{p}$, $\overrightarrow{AB}$ is the direction vector $\vec{d}$. Then, $\overrightarrow{AP} = \vec{p} - \vec{a} = k\overrightarrow{AB} = k\vec{d}$

$$\vec{p} = \vec{a} + k\vec{d},\ k \in \mathbb{R}.$$

- Vector equation of a line:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_2 \end{pmatrix} + k \begin{pmatrix} d_1 \\ d_2 \end{pmatrix},\ k \in \mathbb{R}.$$

- Parametric form:

$$\begin{cases} x = x_1 + kd_1 \\ y = y_1 + kd_2 \end{cases},\ k \in \mathbb{R}.$$

- Cartesian form:

$$\frac{x - x_1}{d_1} = \frac{y - y_1}{d_2}.$$

Proof 3.2.5

$$\begin{cases} x = x_1 + kd_1 \\ y = y_1 + kd_2 \end{cases} \Rightarrow \begin{cases} k = \frac{x - x_1}{d_1} \\ k = \frac{y - y_1}{d_2} \end{cases}.$$

  (a) Cartesian form can be further rearranged to slope-intercept form

$$\frac{x - x_1}{d_1} = \frac{y - y_1}{d_2}$$
$$\frac{d_2}{d_1}(x - x_1) = y - y_1$$
$$y = \frac{d_2}{d_1}(x - x_1) + y_1,$$

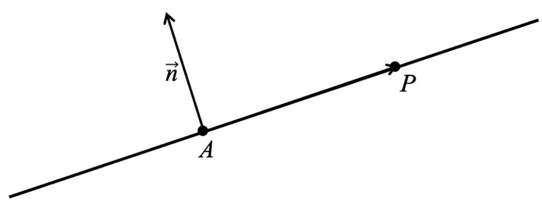
  where $\frac{d_2}{d_1}$ is the slope.

(b) Another way of interpretation:

$$\begin{aligned}\overrightarrow{AP} = k\overrightarrow{AB} \Rightarrow \vec{p}-\vec{a} &= k\left(\vec{b}-\vec{a}\right)\\ \vec{p} &= (1-k)\vec{a}+k\vec{b},\ k\in\mathbb{R}.\\ \Rightarrow \begin{pmatrix} x\\ y\end{pmatrix} &= (1-k)\begin{pmatrix} x_1\\ y_1\end{pmatrix}+k\begin{pmatrix} x_2\\ y_2\end{pmatrix},\ k\in\mathbb{R}.\\ \Rightarrow &\begin{cases} x=(1-k)x_1+kx_2=x_1+k(x_2-x_1)\\ y=(1-k)y_1+ky_2=y_1+k(y_2-y_1)\end{cases},\ k\in\mathbb{R}.\\ \Rightarrow &\begin{cases} k=\frac{x-x_1}{x_2-x_1}\\ y=y_1+k(y_2-y_1)\end{cases}\\ \Rightarrow\ y &= y_1+\frac{x-x_1}{x_2-x_1}(y_2-y_1)\\ &= \frac{y_2-y_1}{x_2-x_1}(x-x_1)+y_1.\end{aligned}$$

4. Orthogonal / Perpendicular vector of a line.

- There is one and only one line in the plane that is perpendicular to a given line at a particular point on that line.
- Normal Vector:



Definition 3.2.2 A **normal vector** is perpendicular or **orthogonal** to any vector on the lines.

$$\text{i.e.,} \vec{n}\cdot\overrightarrow{AP} = 0.$$

Theorem 3.2.2

$$\vec{n}\cdot(\vec{p}-\vec{a}) = 0 \ \Rightarrow\ \vec{n}\cdot\vec{p} = \vec{n}\cdot\vec{a}.$$

- If the direction vector $\vec{d} = \begin{pmatrix} d_1\\ d_2\end{pmatrix}$, then one possible normal vector would be $\vec{n} = \begin{pmatrix} d_2\\ -d_1\end{pmatrix}$ or any other vectors parallel to it.

- The vector form:

$$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} d_2 \\ -d_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \cdot \begin{pmatrix} d_2 \\ -d_1 \end{pmatrix}$$

$$\Rightarrow xd_2 - yd_1 = x_1 d_2 - y_1 d_1$$

$$(x - x_1)d_2 = yd_1 - y_1 d_1$$

$$\therefore y = \frac{d_2}{d_1}(x - x_1) + y_1.$$

5. Direction vectors:

   - **Parallel lines** have **collinear** direction vectors.
   - **Perpendicular lines** have **orthogonal** direction vectors, such that the scalar product is equal to 0.

6. Vector equation of lines in 3-D spaces:

   - $$\vec{r} = \vec{a} + \lambda \vec{d},\ \lambda \in \mathbb{R}.$$

     $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

   - The parametric form:

     $$\begin{cases} x = a_1 + \lambda d_1 \\ y = a_2 + \lambda d_2 \\ z = a_3 + \lambda d_3 \end{cases},\ \lambda \in \mathbb{R}.$$

   - The Cartesian form:

     $$\frac{x - a_1}{d_1} = \frac{y - a_2}{d_2} = \frac{z - a_3}{d_3}.$$

7. Two lines:

   - 2-D spaces: two distinctive lines can either be parallel or they can intersect.
   - 3-D spaces:
     (a) Lines are parallel.
     (b) Lines intersect at one common points.
     (c) Lines are **skewed** (do not intersect and are not parallel).

3.2.4 Vector Product and Properties

1. The vector product is an operation that takes two vectors and results in another vector.

   - Definition

Definition 3.2.3 Given the two vectors and their components, $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then the **vector product** is given by:

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

- The vector product of two vectors is another vector that is perpendicular to both vectors.
- Magnitude of the vector product:

  Theorem 3.2.3 The magnitude of the vector product is given by the formula

  $$\left|\vec{a} \times \vec{b}\right| = |\vec{a}| \cdot \left|\vec{b}\right| \cdot \sin\theta,$$

  where θ is the angle between those two vectors. If $\vec{a} \times \vec{b} = 0$, then $\vec{a}$ and $\vec{b}$ are parallel/-collinear.

- The geometrical definition of cross product (vector product):

  Theorem 3.2.4 Given two vectors $\vec{a}$ and $\vec{b}$, then the vector product is given by

  $$\vec{a} \times \vec{b} = \left(|\vec{a}| \left|\vec{b}\right| \sin\theta\right) \widehat{n},$$

  where $\widehat{n}$ is the unit vector whose direction is given by the right-hand screw rule to both $\vec{a}$ and $\vec{b}$ and the vectors $\vec{a}$, $\vec{b}$, and $\widehat{n}$ follows the right-hand rule.

- Geometrical meaning of the magnitude of the vector product:
  It is equal to the area of the parallelogram enclosed by those two vectors.

2. Properties of the vector product:

   - $$\vec{a} \times \vec{b} = -\left(\vec{b} \times \vec{a}\right)$$
   - $$\left(\vec{a} \times \vec{b}\right) \times \vec{c} = \vec{a} \times \left(\vec{b} \times \vec{c}\right)$$
   - $$\lambda\left(\vec{a} \times \vec{b}\right) = (\lambda\vec{a}) \times \vec{b} = \vec{a} \times \left(\lambda\vec{b}\right), \ \lambda \in \mathbb{R}$$
   - $$\left(\vec{a} + \vec{b}\right) \times \vec{c} = (\vec{a} \times \vec{c}) + \left(\vec{b} \times \vec{c}\right).$$

3. Mixed product:

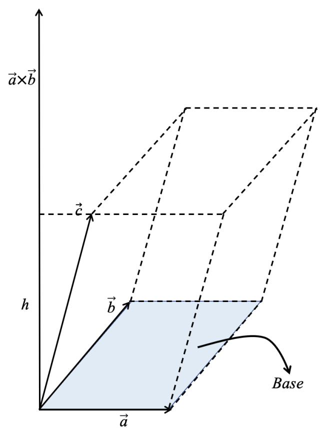
   - An operation with three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ combining both the vector and scalar product is called a **mixed product**:
     $$\left(\vec{a} \times \vec{b}\right) \cdot \vec{c}.$$

- Given $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, and $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$, the mixed product is given by:

$$\left(\vec{a} \times \vec{b}\right) \cdot \vec{c} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$
$$= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$$

- Geometric meaning of mixed products:
  The volume of a parallelepiped formed by three non-coplanar vectors, $\vec{a}$, $\vec{b}$, and $\vec{c}$ is given by:

$$V = \left|\left(\vec{a} \times \vec{b}\right) \cdot \vec{c}\right|.$$



Proof 3.2.6

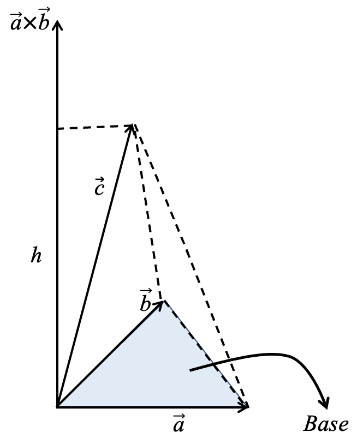
$$V = \text{Base} \times h$$

Base=magnitude of cross product of $\vec{a}$ and $\vec{b}$.
$=$ perpendicular projection of $\vec{c}$ to $\vec{a} \times \vec{b}$.

$$\therefore V = \text{Base} \times h = \left|\vec{a} \times \vec{b}\right| \cdot |\vec{c}| \cdot |\cos\theta| = \left|\left(\vec{a} \times \vec{b}\right) \cdot \vec{c}\right|$$

- Three or more vectors are said to be coplanar if they lie in the same plane.
- Using mixed product to find the volume of a triangular pyramid:

$$V = \frac{1}{6}\left|\left(\vec{a} \times \vec{b}\right) \cdot \vec{c}\right|.$$



Proof 3.2.7 Since the base is not a parallelogram but a triangle, that is half an area of the parallelogram, we multiply $\frac{1}{2}$ in front of the expression of the cross product.

$$\text{Base} = \frac{1}{2}\left|\vec{a}\times\vec{b}\right|.$$

The volume of a pyramid is $\frac{1}{3}$ of the product of the base and the height.

$$\therefore V = \frac{1}{3}\text{Base}\cdot h = \frac{1}{3}\cdot\frac{1}{2}\left|\vec{a}\times\vec{b}\right|\left|\vec{c}\right|\left|\cos\theta\right| = \frac{1}{6}\left|\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right|$$

4. Proving vector product using matrix.

Proof 3.2.8 Let $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. Convert into a 3×3 matrix: $\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$.
Find the determinant: $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i}(a_2b_3 - a_3b_2) - \vec{j}(a_1b_3 - a_3b_1) + \vec{k}(a_1b_2 - a_2b_1)$

$$\Rightarrow \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

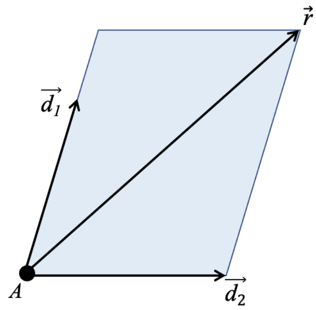
3.2.5 Vector Equation of a Plane

1. A plane is uniquely determined by three points (or a line and a point outside the line).
   $\rightarrow$ A plane can also be determined by two intersecting lines and a point outside the lines.

2. Vector equation of a plane:

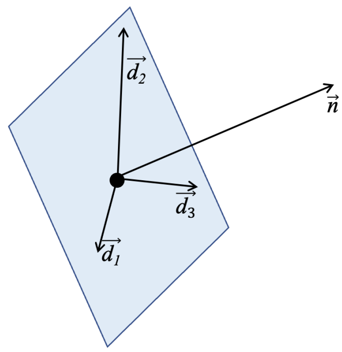
$$\vec{r} = \vec{a} + \lambda\vec{d}_1 + \mu\vec{d}_2,\ \lambda,\mu \in \mathbb{R}.$$

   where $\vec{d}_1$ and $\vec{d}_2$ are direction vectors, and $\vec{a}$ is the position vector.



3. The scalar product form:

   - **Normal vector** is a vector that is perpendicular to every line in the plane.



   - All planes with the same normal vector are parallel to each other.
   - If R is any other point on the plane, then $\overrightarrow{AR}$ lies in the plane, and it is perpendicular to the normal vector $\vec{n}$.

   Theorem 3.2.5

$$\overrightarrow{AR} \cdot \vec{n} = 0 \ \Rightarrow \ (\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

$$\therefore \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

   where $\vec{a}$ is the position vector, and $\vec{n}$ is the normal vector.

4. The Cartesian equation of a plane:

$$n_1x + n_2y + n_3z = d, \text{ where } n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}, \ d = \vec{a} \cdot \vec{n}.$$

Proof 3.2.9

$$\vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}, \ d = \vec{a} \cdot \vec{n}, \ \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The scalar product form converts to:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \vec{a} \cdot \vec{n}$$

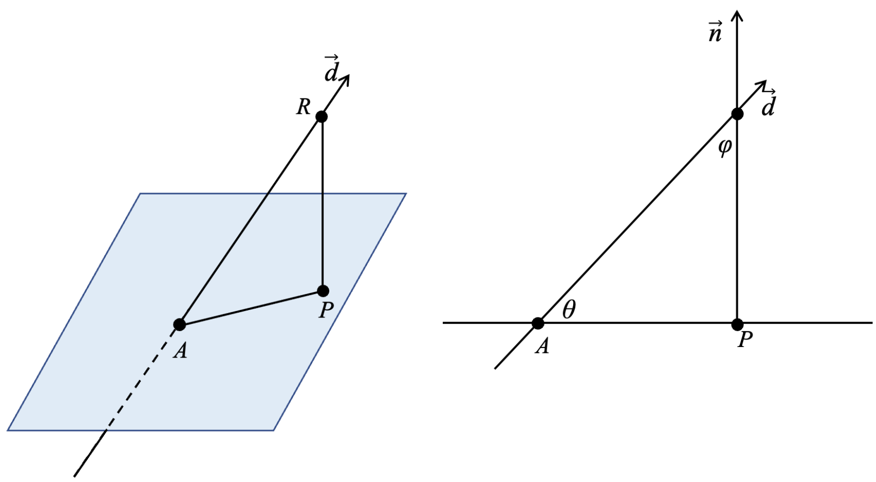
$$\Rightarrow n_1 x + n_2 y + n_3 z = d.$$

5. A plane with the vector equation $\vec{r} = \vec{a} + \lambda \vec{d_1} + \mu \vec{d_2}$ has a normal vector $\vec{n} = \vec{d_1} \times \vec{d_2}$.

3.2.6 Lines, Planes, and Angles

1. Angles and intersections between lines and planes:

   - When a line intersects a plane, the angle between them is defined as the smallest possible angle that the line makes with any of the lines in the plane.



     (a) $\overrightarrow{AR}$: the direction vector of the line, $\vec{d}$.
     (b) Point P is the projection of point R onto the plane. $\overrightarrow{AP}$ is the shadow of $\overrightarrow{AR}$ on the plane.
     (c) $\overrightarrow{PR}$ is in the direction of $\vec{n}$ since it is perpendicular to the plane.
     (d) φ is the angle between $\vec{n}$ and $\vec{d}$.
     (e)

$$\theta = 90^\circ - \varphi, \ \cos\varphi = \frac{\left|\vec{n} \cdot \vec{d}\right|}{|\vec{n}| \left|\vec{d}\right|}.$$

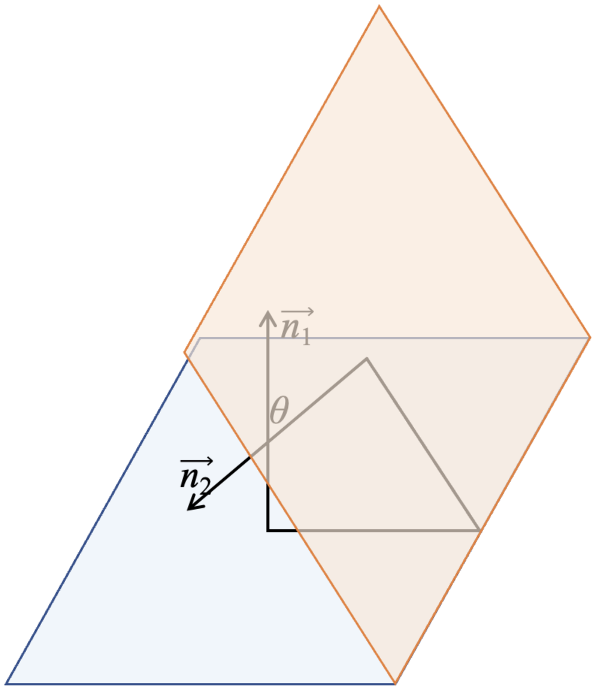
   - A line that is not parallel to a plane intersects a plane at one point. The coordinates of this point of intersection satisfies both the equation of the line and the equation of the plane.

2. Relationship of two planes:

- Two planes can either intersect at a line or they can be parallel.
- When two planes are parallel, their normal vectors are collinear; otherwise they intersect at a line.

3. Angles between two planes:

- The angle between two planes is **the angle between their normal vectors**.
- $$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|}.$$



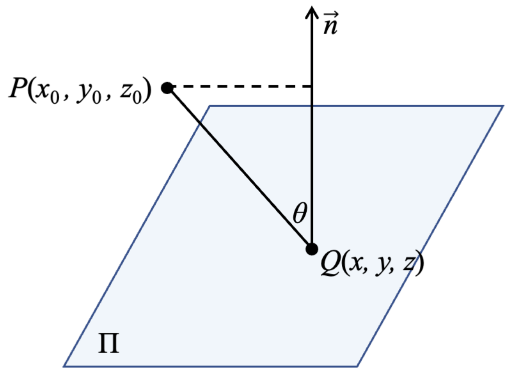
4. Two non-parallel planes intersect along a line. The equation of this line is formed by treating the Cartesian equation of two planes as simultaneous equations and finding the general solution.

5. Distance between a point and a plane.

- The distance, d, between a point $P(x_0, y_0, z_0)$, and a plane with equation $Ax+By+Cz=D$ where $\vec{n}=A\vec{i}+B\vec{j}+C\vec{k}$, is given by:

$$d = \frac{|Ax_0+By_0+Cz_0-D|}{\sqrt{A^2+B^2+C^2}}.$$

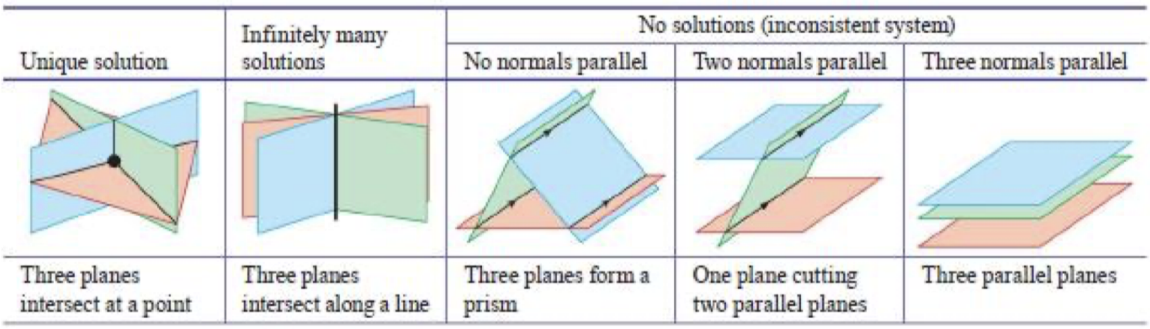
- Proof:



Proof 3.2.10 Let $Q(x,y,z)$ be any point on the plane Π.
The distance, d, is the projection of the distance of point P to the plane on the normal vector, $\vec{n}$.

$$d = \left|\overrightarrow{QP}\right| \cdot |\cos\theta| = \left|\overrightarrow{QP}\right| \cdot \frac{\overrightarrow{QP} \cdot \vec{n}}{\left|\overrightarrow{QP}\right| \cdot |\vec{n}|}$$
$$= \frac{\overrightarrow{QP} \cdot \vec{n}}{|\vec{n}|} = \frac{|\langle A,B,C\rangle \cdot \langle (x_0 - x),(y_0 - y),(z_0 - z)\rangle|}{\sqrt{A^2+B^2+C^2}}$$
$$= \frac{|Ax_0 + By_0 + Cz_0 - (Ax + By + Cz)|}{\sqrt{A^2+B^2+C^2}}$$
$$= \frac{Ax_0 + Bx_0 + Cz_0 - D}{\sqrt{A^2+B^2+C^2}}$$

6. Intersection of three points:



Unique solution	Infinitely many solutions	No solutions (inconsistent system)		
		No normals parallel	Two normals parallel	Three normals parallel
Three planes intersect at a point	Three planes intersect along a line	Three planes form a prism	One plane cutting two parallel planes	Three parallel planes

- The plane intersect:
  (a) At a point: the system of equations will have a unique solution.
  (b) Alone a line: the system of equations will have infinitely many solutions
- The systems of equations have no solutions:
  (a) No normals are parallel (the planes from a prism)
  (b) 2 normals are parallel or three normal are parallel (the planes are parallel)

4 Statistics and Probability

4.1 An Introduction to Statistics

1. Statistical inferences: when the sample data is well chosen and described, an analysis will allow us to draw conclusions about the population based on this sample.

   - **Discrete data** is data that can be counted; it gives the number of times something occurs or the number of items that exists.
   - **Continuous data** is data that is measured; however, the values of the actual data cannot be determined exactly, and the data may be limited to a range.

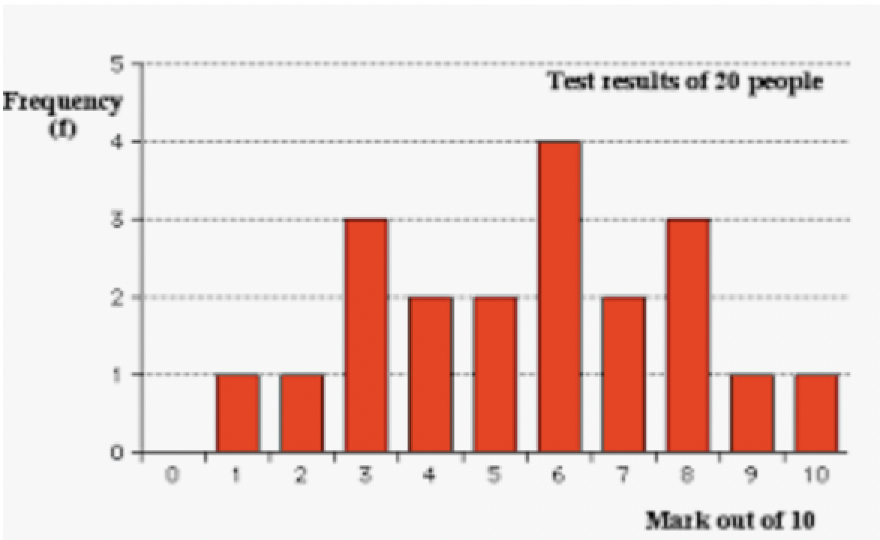
2. Reliability and Bias.

3. Data Sampling (Sampling Methods):

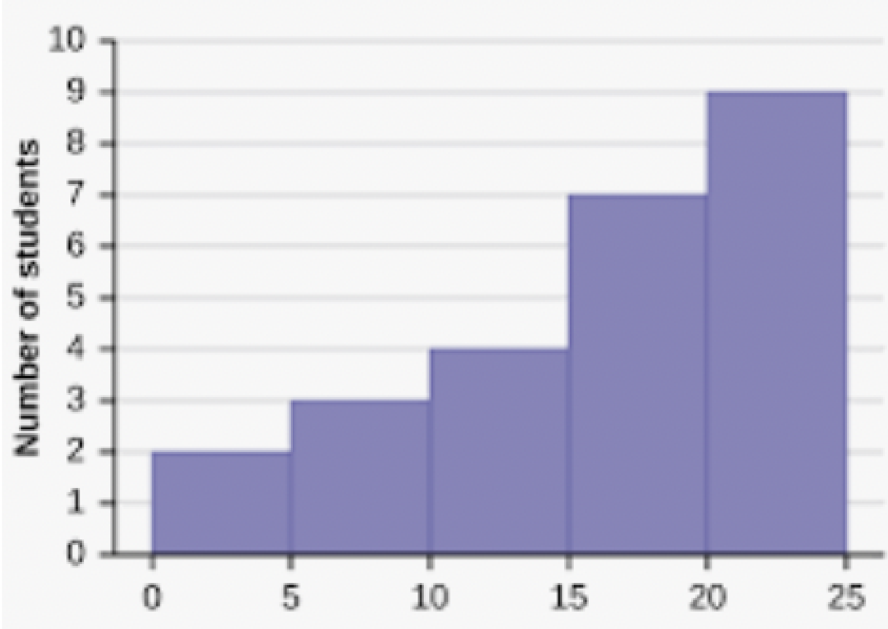
   - **Simple**: Achieving randomness by a simple, completely random process.
   - **Convenience**: Choosing a sample based on how easy it is to find the data.
   - **Systematic**: If data is listed, selecting a random starting point and then choosing the rest of the sample at a consistent interval in the list.
   - **Quota**: Choosing a sample that is only comprised of members of the population that fit certain characteristics.
   - **Satisfied**: Choosing a random sample in a way that the population of certain characteristics matches the proportion of those characteristics in the population.

4. Grouped Data:

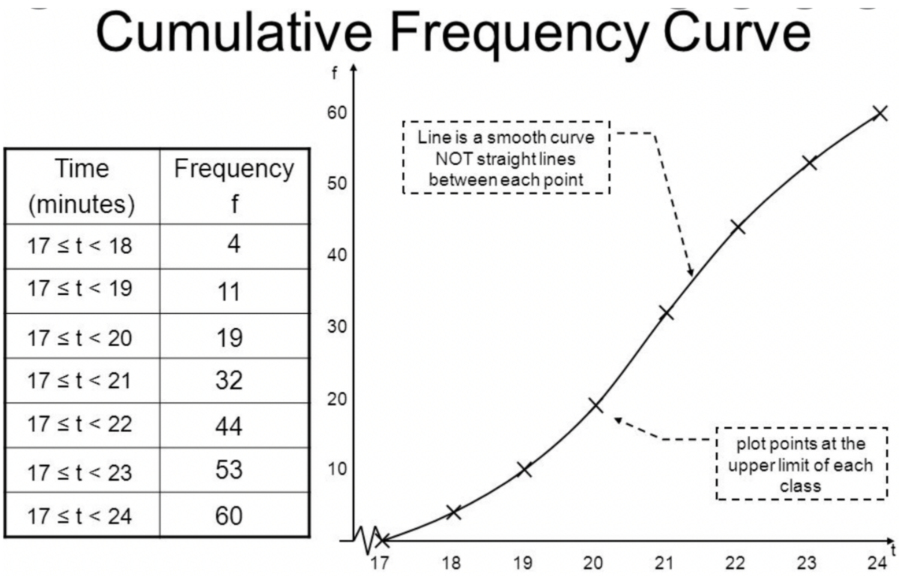
   - Key terms: intervals/classes, frequency distribution table, frequency diagram.
   - Frequency diagram of discrete data:



   - Frequency diagram of continuous data - **Histogram**



- Cumulative frequency table and cumulative frequency graph:



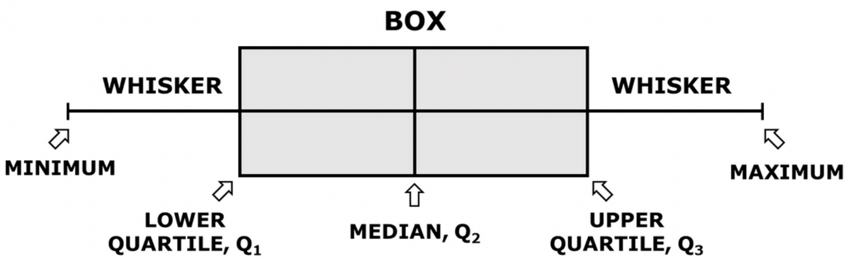
Time (minutes)	Frequency f
17 ≤ t < 18	4
17 ≤ t < 19	11
17 ≤ t < 20	19
17 ≤ t < 21	32
17 ≤ t < 22	44
17 ≤ t < 23	53
17 ≤ t < 24	60

- For data grouped into intervals or classes, we can identify:
  (a) Mid-interval values
  (b) Interval width
  (c) Lower interval boundaries
  (d) Higher interval boundaries
  (e) Modal class: the class with the highest frequency

5. Quartiles:

- Minimum: the lowest value
- Q_1: the 25th percentile
- Median (Q_2): the 50th percentile

- Q_3: the 75th percentile
- Maximum: the highest value
- $$\text{Interquartile Range (IQR)} = Q_3 - Q_1.$$
- Box-and-whisker plots: $\rightarrow$ spread of data:



6. Normal distribution, negatively skewed, and positively skewed:
7. Normal distribution: mean = mode = median
8. Positively skewed: median < mean
9. Negatively skewed: median > mean
10. Outliers:
$$\begin{aligned} &\text{Ourlier} < Q_1 - 1.5\text{IQR} \\ \text{OR} \quad &\text{Outlier} > Q_3 + 1.5\text{IQR}. \end{aligned}$$
11. Measuring central tendency:
    - **Definition 4.1.1** **Mode**: the value with the greatest frequency.
    - **Definition 4.1.2** **Median**(m): the middle value when the data is arranged in order.
    - **Definition 4.1.3** **Mean**(μ): the arithmetic mean, the sum of the numerical data divided by the number of data points.
    $$\mu = \frac{\sum_{i=1}^{n} x_i}{n}.$$
12. Measures of dispersion:
    - **Definition 4.1.4** Range:
    $$\text{Range} = x_{\max} - x_{\min}.$$
    - **Definition 4.1.5** Interquartile Range (IQR):
    $$\text{IQR} = Q_3 - Q_1.$$
    - **Definition 4.1.6** Variance σ^2 and standard deviation σ:
    $$\begin{aligned} \sigma^2 &= \frac{\sum_{i=1}^{k} f_i (x_i - \mu)^2}{n} = \frac{\sum_{i=1}^{k} f_i x_i^2}{n} - \mu^2 \\ \sigma &= \sqrt{\sigma^2} \end{aligned}$$

Proof 4.1.1

$$\begin{aligned}\sigma^2 = \frac{\sum_{i=1}^{k} f_i (x_i - \mu)^2}{n} &= \frac{\sum_{i=1}^{k} f_i x_i^2}{n} - 2\mu \frac{\sum_{i=1}^{k} f_i x_i}{n} + \mu^2 \frac{\sum_{i=1}^{k} f_i}{n} \\ &= \frac{\sum_{i=1}^{k} f_i x_i^2}{n} - 2\mu^2 + \mu^2 \\ &= \frac{\sum_{i=1}^{k} f_i x_i^2}{n} - \mu^2\end{aligned}$$

4.2 Linear Correlation of Bivariate Data

1. Scatter Plot
2. Linear correlation:
   - **Positive linear correlation**: the line as a general upward trend.
   - Negative correlation.
   - Quadratic trend
   - Exponential trend
3. Estimate a line of best fit:

   Theorem 4.2.1 The line must pass through the mean point of the data set $(\bar{x}, \bar{y})$, where $\bar{x}$ is the mean of x and $\bar{y}$ is the mean of y.

4. Pearson's correlation coefficient (r):
   - Range: $-1 \leq r \leq 1$
     (a) $r = -1$: perfect negative linear correlation
     (b) $r = 1$: perfect positive linear correlation
     (c) $r = 0$: no linear correlation
     (d) $0.7 < |r| \leq 1$: strong linear correlation
     (e) $0.3 < |r| \leq 0.7$: weak to moderate linear correlation
   - Formula:

$$r = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sqrt{\sum (x - \bar{x})^2 \sum (y - \bar{y})^2}}$$

$$r = \frac{\sum xy - n\bar{x}\,\bar{y}}{\sqrt{\sum x^2 - n\bar{x}^2}\sqrt{\sum y^2 - b\bar{y}^2}}$$

5. Prediction using the least square linear regression:
   - The dangers of extrapolation.
   - We cannot always reliably make a prediction of x from a value of y, using a y on x line.

4.3 Probability and Expected Outcomes

1. **Definition 4.3.1 Random Events**: although we don't know the exact result in advance, we do know the set of all possible results.

   - The set of all possible results is called the **sample space** (U).
   - The number of all possible outcomes that make up the sample space is denoted as $n(U)$.
   - Any subset of a sample space is called an **event**. The event consists of one or more outcomes.
   - Each time an experiment is repeated, it is considered a **trial** of the experiment.

2. **Experimental probability** (relative frequency) is found by repeating an experiment a number of times and counting the number of times that particular outcome occurs.

3. **Definition 4.3.2 Mutually exclusive**: two events that do not share any outcomes $\rightarrow$ They cannot occur together.

4. **Definition 4.3.3 Theoretical probability**: When running an experiment for N trials that results in an event A occurring $n(A)$ times, the probability of A happening, $\mathrm{P}(A)$ is

$$\mathrm{P}(A) = \lim_{N\to\infty} \frac{n(A)}{N}.$$

   If an experiment has equally likely outcomes, the probability of event A occurring is defined as

$$\mathrm{P}(A) = \frac{n(A)}{n(U)} = \frac{\text{Number of outcomes in which A occurs}}{\text{Total number of outcomes in the sample space}}.$$

5. Axioms:

   Axiom 4.3.1 For any event A,

$$0 \leq \mathrm{P}(A) \leq 1.$$

   Axiom 4.3.2 The probability of nothing (o) occurring is zero:

$$\mathrm{P}(o) = 0.$$

   The probability of one of all the outcomes in the sample space occurring is one:

$$\mathrm{P}(U) = 1.$$

   Axiom 4.3.3 If A and B are $\in U$ and are mutually exclusive, then the probability of either A or B happening is

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B).$$

   Axiom 4.3.4 If $\mathrm{P}(A')$ is the probability of event A not happening,

$$\mathrm{P}(A') = 1 - \mathrm{P}(A),$$

   where $\mathrm{P}(A)$ is the probability of event A occurring.

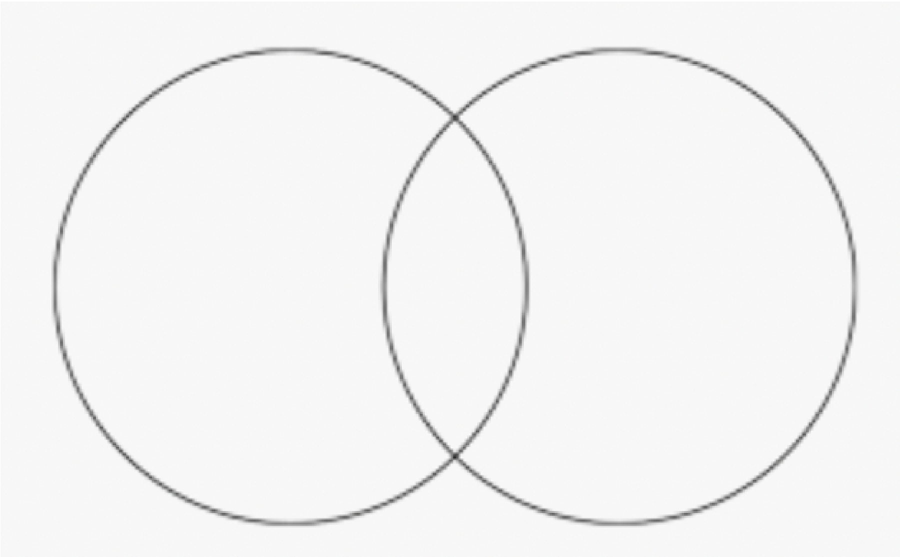
6. **Definition 4.3.4** Expectation: The formula for expected number of members of group A in a sample of size n is

$$nP(A).$$

If choosing n from the population, $nP(A)$ gives an approximation of how many of them would be members of group A.

4.4 Probability Calculations

1. **Definition 4.4.1** A **Venn diagram** is a model illustrating two or more sets of data using overlapping circles to show elements of each set.



- **Definition 4.4.2** The probability of event A and B occurring at the same time:

$$P(A \cap B)$$

- **Definition 4.4.3** The probability of event A or B occurring (A union B):

$$P(A \cup B)$$

- **Theorem 4.4.1** If $P(A \cup B) = \varnothing$, then A and B are **mutually exclusive**.

2. The tree diagram

3. **Theorem 4.4.2** The probability of a union:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof 4.4.1

$$n(A \cup B) = n(A) + n(B) - n(A \cap B),$$

$$P(A \cup B) = \frac{n(A \cup B)}{n(U)} = \frac{n(A) + n(B) - n(A \cap B)}{n(U)}$$
$$= P(A) + P(B) - P(A \cap B).$$

4. Conditional probability:

   - Conditional probability shrinks the sample space and, therefore, increases the probability of an event occurring, unless the given information renders the event impossible.
   - **Theorem 4.4.3**
   $$P(A|B) = \frac{P(A \cap B)}{P(B)},$$
   where $P(A|B)$ is read as probability of event A occurring, given event B occurring.

     Proof 4.4.2 If we know that B has occurred, the sample space now contains all the elements of B but no more. Now, we select events from the sample space that falls in A:
   $$\begin{aligned} P(A|B) &= \frac{n(A \cap B)}{n(B)} \\ &= \frac{\frac{n(A \cap B)}{n(U)}}{\frac{n(B)}{n(U)}} = \frac{P(A \cap B)}{P(B)}. \end{aligned}$$

5. The probability of independent events:

   - **Definition 4.4.4** A and B are **independent events** if the occurrence of A has no effect on the probability of B occurring.
   $$P(B|A) = P(B) \quad \text{OR} \quad P(A|B) = P(A).$$
   - **Theorem 4.4.4** A and B are independent if and only if
   $$P(A \cap B) = P(A) \times P(B).$$

6. **Theorem 4.4.5** Probability of union when A and $B$are mutually exclusive:
$$P(A \cup B) = P(A) + P(B).$$

4.5 Discrete Random Variables

1. **Definition 4.5.1** If a random variable can take exactly N values, each of which corresponds to a unique outcome in the sample space of an experiment, then this variable is a **discrete random variable**.

   - The probability that x takes on any of the N values is written as $P(X = x)$
   - A **probability distribution** is a combination of the sample space of a random experiment with the probabilities of each of the events in the sample space.
   - PDF: probability distribution function
   - **Axiom 4.5.1** For each value $\{x_i\}$ of the random variable X, we have that
   $$0 \leq P(X = x_i) \leq 1.$$

- **Axiom 4.5.2** As X may take on any of the N values of the sample space, the sum of the probabilities of each of them as an outcome of the experiment must equal one:

$$\sum_{i=1}^{N} \mathrm{P}(X = x_i) = \mathrm{P}(X = x_1) + \mathrm{P}(X = x_2) + \mathrm{P}(X = x_3) + \cdots + \mathrm{P}(X = x_N) = 1.$$

- **Definition 4.5.2** **Well-defined probability distribution**: a probability distribution that satisfies both

$$\begin{cases} 0 \leq \mathrm{P}(X = x_i) \leq 1 \\ \sum_{i=1}^{N} \mathrm{P}(X = x_i) = 1 \end{cases}$$

2. Calculating expected value:

- **Definition 4.5.3** The **expected value** (expected mean) is the value you would expect to obtain on average if you performed an experiment many times.
- The expected value of a random variable X with a probability distribution function $\mathrm{P}(X = x)$ is written as $\mathrm{E}(X)$ or μ.
- Formula:

$$\begin{aligned} \mathrm{E}(X) = \mu = \sum_{i=1}^{N} x_i \mathrm{P}(X = x_i) \\ = x_1 \mathrm{P}(X = x_1) + x_2 \mathrm{P}(X = x_2) + x_3 \mathrm{P}(X = x_3) + \cdots + x_N \mathrm{P}(X = x_N) \end{aligned}$$

- Application: fairness of a game
  A fair game is perfectly balanced so that the expected mean pay-out is 0 for both players.

4.6 The Binomial Distribution

1. The binomial distribution discrete the probability distribution of different outcomes of repeated binary events.

- Since there are only two outcomes, we use p to represent the probability of success, meaning the probability that the event we are looking for occurs.
- We use $X \sim$ to denote a random variable X that is distributed in a certain way.
- $X \sim \mathrm{B}(n, p)$ represents the binomial distribution,
  where $mathrmB$ stands for binomial, n represents the number of trials in the binomial experiment, and p is the probability of success.
- The probability of failure is q:

$$q = 1 - p.$$

- **Theorem 4.6.1** Expected value of a binomial distribution:

$$\mathrm{E}(X) = np.$$

- **Theorem 4.6.2** Variance of a binomial distribution:

$$\mathrm{Var}(X) = np(1-p) = npq.$$

2. Probabilities within the binomial distribution:

   Theorem 4.6.3 A random variable that is binomially distributed and takes on the value of the number of success in n trials is written as $X \sim \mathrm{B}(n,p)$, where p is the probability of success in one trial. Then, the probability that X takes on an actual value of x successes is given by:

$$X \sim \mathrm{B}(n,p) \Rightarrow \mathrm{P}(X=x) = \binom{n}{x} p^x (1-p)^{n-x},\ x = 0,1,2,3,\cdots,n.$$

3. Finding binomial probabilities using GDC:

- **Binomial PDF** returns a specific value of $\mathrm{P}(X=x)$
- **Binomial CDP** (cumulative distribution function) gives the value of

$$\mathrm{P}(0 \le X \le \text{upper bound}).$$

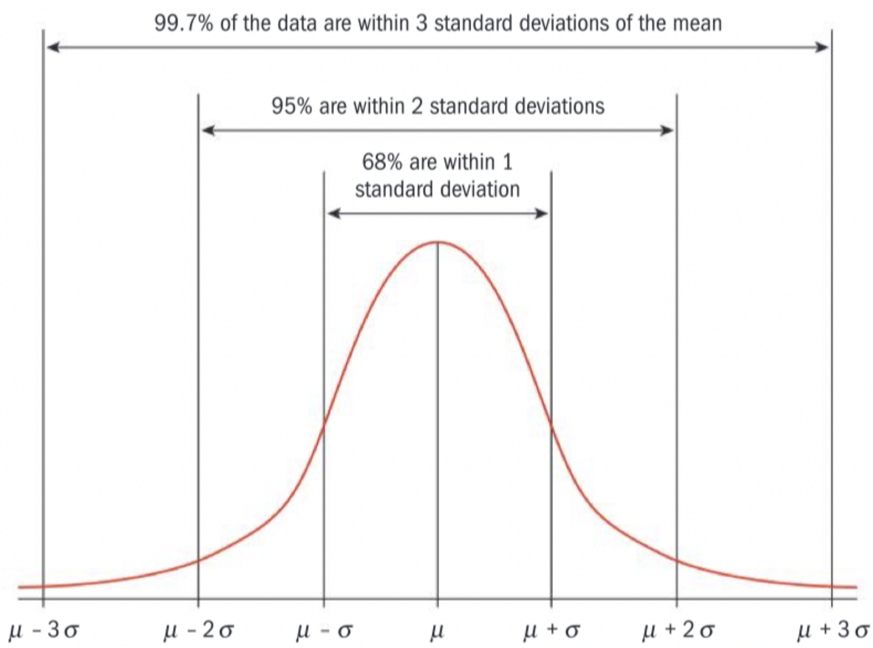
- **Theorem 4.6.4**

$$\mathrm{P}(a \le X \le b) = \mathrm{P}(0 \le X \le b) - \mathrm{P}(0 \le X \le a-1).$$

4.7 The Normal Distribution and Curve

1. The normal (bell) curve:

- The normal distribution is modeled graphically with a **normal curve**:



- The normal curve always has its highest point in the center, and that point is the mean (μ), median, and mode.
- The axis is often labeled using the standard deviation σ.
- The normal distribution deals with continuous random variables.

2. Expressing the normal distribution algebraically:

- Normal distribution is denoted as

$$X \sim \mathrm{N}(\mu, \sigma^2),$$

where N means normal distribution, μ is the mean, and σ is the standard deviation.
- Normal probability density function:
  (a) A probability density function is the equation of a curve that has the probabilities as the area underneath.
  (b) **Definition 4.7.1** For $X \sim \mathrm{N}(\mu, \sigma^2)$,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \text{ where } -\infty < x < \infty.$$

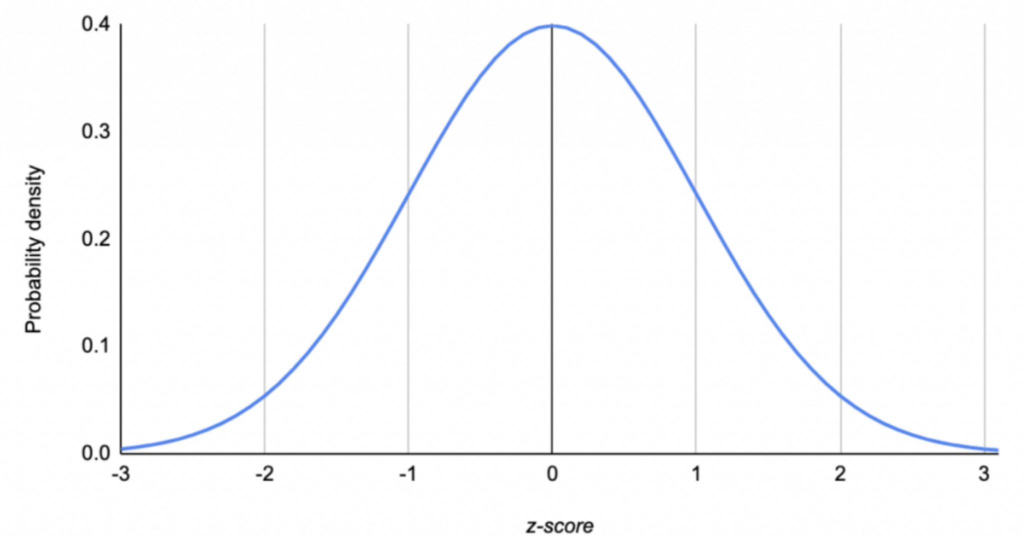
$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \mathrm{d}x = 1.$$

3. Calculating normal distributed probabilities with GDC:

- Normal PDF and Normal CDF.
- Inverse Normal Function.

4. Standard Normal Distribution:

- Transform $X \sim \mathrm{N}(\mu, \sigma^2)$ into $Z \sim \mathrm{N}(0, 1)$:



- Z-score:

$$z = \frac{x-\mu}{\sigma}.$$

- Finding probabilities with z-scores:

  (a)

$$P(-1 < z < 1) \approx 0.68$$
$$P(-2 < z < 2) \approx 0.95$$
$$P(-3 < z < 3) \approx 0.997$$

  (b) Normal CDP on GDC

- Using z-score to find μ and σ.

4.8 Probability Density Function (PDF)

1. For a **continuous random variable** X with **probability density function** $f(x)$, it holds

$$\begin{cases} f(x) \geq 0 & \text{i.e., the function is non-negative} \\ \int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1 & \text{i.e., the total area under the curve is 1} \end{cases}.$$

2. The probability that X takes values between a and b is

$$P(a \leq X \leq b) = \int_a^b f(x)\mathrm{d}x$$

   N.B.: $P(a \leq X \leq b) = P(a < X < b)$ because $P(X = a) = 0$.

3. The **mean** μ, or the **expected value** $E(X)$, is defined by

$$\mu = E(x) = \int_{-\infty}^{\infty} x f(x)\mathrm{d}x.$$

4. The **Variance** $\mathrm{Var}(X)$ is defined by

$$\mathrm{Var}(X) = E(X-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\mathrm{d}x.$$

   An equivalent and more practical definition is

$$\mathrm{Var}(X) = E(X^2) - \mu^2, \text{ where } E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\mathrm{d}x.$$

5. Mode: the value of x where $f(x)$ has its maximum.

6. Median: The value of m where $P(X \leq m) = 0.5$

   Example 4.8.1

$$\text{Let } f(x) = \begin{cases} f_1(x), a \leq x \leq b \\ f_2(x), b < x \leq c \end{cases}, \text{ we check } \int_a^b f_1(x)\mathrm{d}x = A:$$

(a) If $A > 0.5$, the median $m \in [a,b]$, solve

$$\int_a^m f_1(x)\mathrm{d}x = 0.5.$$

(b) If $A < 0.5$, the median $m \in [b,c]$, solve

$$\int_m^c f_2(x)\mathrm{d}x = 0.5.$$

7. Quartiles:

- The **lower quartile Q_1** is defined by $\mathrm{P}(X \leq Q_1) = 0.25$, i.e., solve

$$\int_{-\infty}^{Q_1} f(x)\mathrm{d}x = 0.25.$$

- The **upper quartile Q_3** is defined by $\mathrm{P}(X \leq Q_3) = 0.75$, i.e., solve

$$\int_{-\infty}^{Q_3} f(x)\mathrm{d}x = 0.75.$$

8. The effects of linear transformations on the random variable X:

- $$\mathrm{E}(aX+b) = a\mathrm{E}(X)+b.$$

- $$\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X).$$

4.9 The Bayes' Theorem

1. Bayes' Theorem:

Theorem 4.9.1 Bayes' Theorem

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A)\mathrm{P}(B|A)}{\mathrm{P}(B)}.$$

Proof 4.9.1

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)} \Rightarrow \mathrm{P}(A\cap B) = \mathrm{P}(A|B)P(B)$$

$$P(B|A) = \frac{\mathrm{P}(B\cap A)}{\mathrm{P}(A)} \Rightarrow \mathrm{P}(B\cap A) = \mathrm{P}(B|A)P(A)$$

$$\because \mathrm{P}(A\cap B) = \mathrm{P}(B\cap B),$$

$$\therefore \mathrm{P}(A|B)P(B) = \mathrm{P}(B|A)P(A).$$

$$\therefore \mathrm{P}(A|B) = \frac{\mathrm{P}(A)\mathrm{P}(B|A)}{\mathrm{P}(B)}.$$

2. Other versions of the Bayes' theorem:

-
$$P(B) = P(A \cap B) + P(A' \cap B) = P(A)P(B|A) + P(A')P(B|A').$$

- Substitute $P(B)$:
$$P(A|B) = \frac{P(B|A)P(A)}{P(A)P(B|A) + P(A')P(B|A')};$$
$$P(B|A) = \frac{P(A|B)P(B)}{P(B)P(A|B) + P(B')P(A|B')}.$$

3. Bayes' theorem for three events:

$$\begin{aligned} P(B) &= P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) \\ &= P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3). \end{aligned}$$

$$\begin{aligned} \therefore P(A_i|B) &= \frac{P(A_i)P(B|A_i)}{P(B)} \\ &= \frac{P(A_i)P(B|A_i)}{P(A_i)P(B|A_i) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)}. \end{aligned}$$

5 Calculus

5.1 Limits

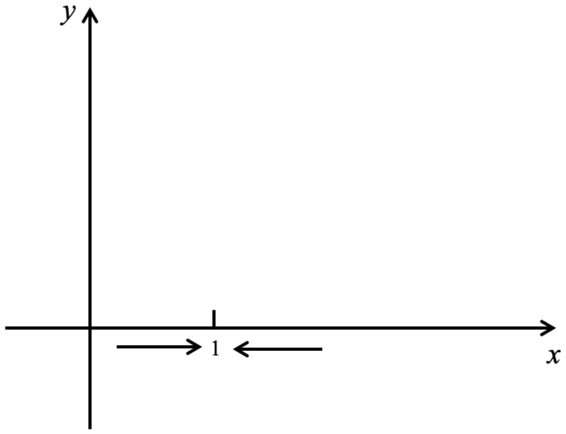
1. Limit

Example 5.1.1

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=2$$

when x is approaching to 1 (it never equals to 1), the value $\frac{x^2-1}{x-1}$ is approaching to 2.

- Left-hand and Right-hand Limit



Example 5.1.2 The left-hand limit of $\frac{x^2-1}{x-1}$ when $x\to 1$ is

$$\lim_{x\to 1^-}\frac{x^2-1}{x-1}=2.$$

The right-hand limit of $\frac{x^2-1}{x-1}$ when $x\to 1$ is

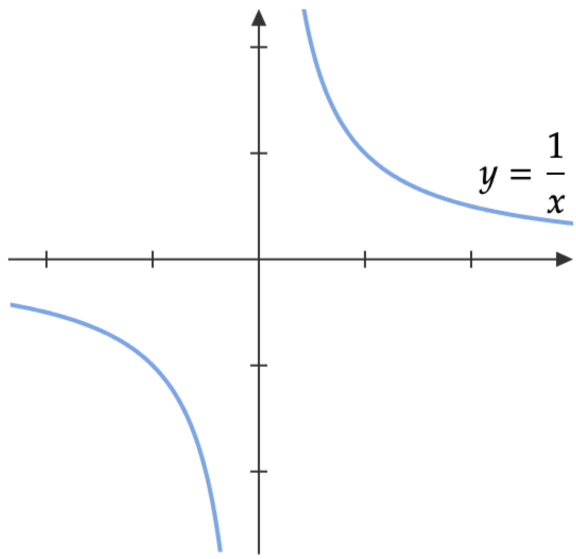
$$\lim_{x\to 1^+}\frac{x^2-1}{x-1}=2.$$

- Only when the left-hand limit and the right-hand limit exist and are the same at the point $x=a$, we say the limit of $f(x)$ exists on $x=a$.

$$\text{i.e., } \lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=c \Rightarrow \lim_{x\to a} f(x)=c,\ c \text{ is a constant} \in \mathbb{R}$$

Otherwise, the limit does not exist on $x=a$ (OR DNE.).

Example 5.1.3 Does $\lim_{x\to 0}\frac{1}{x}$ **exist? How about** $\lim_{x\to\infty}\frac{1}{x}$**?**



- $\lim_{x\to 0} \frac{1}{x}$ does not exist.

$$\because \lim_{x\to 0^+} \frac{1}{x} = +\infty, \ \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

$$\therefore \lim_{x\to 0^+} \frac{1}{x} \neq \lim_{x\to 0^-} \frac{1}{x} \Rightarrow \text{DNE.}$$

- $\lim_{x\to \infty} \frac{1}{x}$ exists.

$$\because \lim_{x\to +\infty} \frac{1}{x} = 0, \ \lim_{x\to -\infty} \frac{1}{x} = 0$$

$$\therefore \lim_{x\to +\infty} \frac{1}{x} = \lim_{x\to -\infty} \frac{1}{x} \Rightarrow \text{Limit exists.}$$

Definition 5.1.1 **Horizontal Asymptote (H.A.)**:

$$y = \lim_{x\to\infty} f(x) = c$$

- Limit at ∞:

$$\lim_{x\to +\infty} f(x) = \lim_{x\to -\infty} f(x) = c \ \Rightarrow \ \lim_{x\to\infty} f(x) = c.$$

Note: $+\infty$ and $-\infty$ are not exact values; they should be regarded as a concept.

- Limits do not have to equal to the function value.
  Limit and the function value do not have relationships.
- Generally speaking, if $a \in D_f$, $\lim_{x\to a} f(x) = f(a)$.

2. For a rational function $f(x) = \frac{P(x)}{Q(x)}$ where $P(x) = a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m$, and $Q(x) = b_0x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m$:

   - $\lim_{x\to a} f(x) = f(a)$ as long as $Q(a) \neq 0$.
   - $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m}{b_0x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m} \ \Rightarrow$ H.A.

     (a) If $m = n$, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{a_0}{b_0} = \frac{a_0}{b_0}$.

(b) If $m > n$, $\lim_{x\to\infty} f(x)$ DNE.

(c) If $m < n$, $\lim_{x\to\infty} f(x) = 0$.

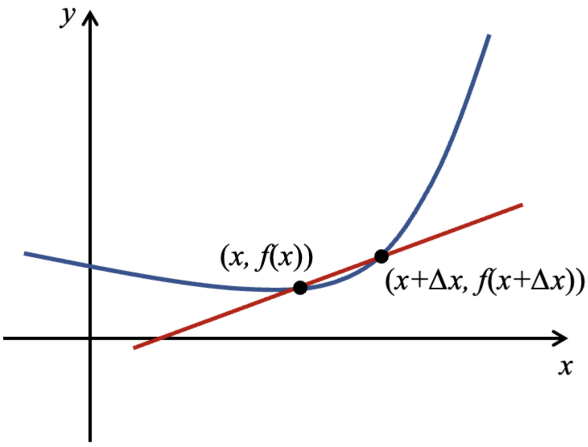
3. Continuity and Discontinuity

Definition 5.1.2 **Continuity**: If the graph of the function does not have any breaks or holes within a certain interval, then the function is continuous within that interval.

Theorem 5.1.1 If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$, then the function f is **continuous** at $x = a$.

5.2 Differentiation and Derivatives

1. Gradient of Secant:



- $$\text{Slope } m = \frac{f(x+\Delta x) - f(x)}{x + \Delta x - x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Definition 5.2.1 **Derivative of a function**:

$$\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \text{ is the derivative of a function, denoted as } \frac{\mathrm{d}y}{\mathrm{d}x} \text{ or } f'(x).$$

- The graphic meaning of derivative is the gradient of tangent of the function.

Example 5.2.1 **By definition, find the derivative of $f(x) = x^2 + 1$ and hence find the gradient of the tangent line when $x = 3$.**

$$\begin{aligned} f'(x) = \lim_{x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} &= \lim_{x\to 0} \frac{\left[(x+\Delta x)^2 + 1\right] - (x^2 + 1)}{\Delta x} \\ &= \lim_{x\to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 1 - x^2 - 1}{\Delta x} \\ &= \lim_{x\to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\ &= \lim_{x\to 0} (2x + \Delta x) \\ &= 2x. \end{aligned}$$

At $x = 3$, $f'(3) = 2 \times 3 = 6$. The gradient is 6.

2. Derivative of x^n

Theorem 5.2.1 If $f(x) = x^n$, then

$$f'(x) = nx^{n-1}, \text{ for any } n \in \mathbb{R}.$$

Note: The derivative of any constant is 0.

Example 5.2.2

$$f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = (-1)x^{-1-1} = -x^{-2};$$
$$f(x) = \sqrt{x} = x^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}};$$
$$f(x) = c = cx^0 \Rightarrow f'(x) = 0 \times cx^{0-1} = 0.$$

3. Rules of Differentiation:
Name $f(x)$ and $g(x)$ as two functions with derivatives of $f'(x)$ and $g'(x)$, respectively. Then

$$(cf(x))' = cf'(x)$$
$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$

4. More Derivatives:

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\ln x$	$\frac{1}{x}$
e^x	e^x

5. Differentiability:

Definition 5.2.2 A function has to be **continuous** and **no sharp turning point** to be **differentiable**.
Note: Smooth turning point on the graph is allowed.

6. More Rules of Differentiation:

Theorem 5.2.2 Let $f(x)$ and $g(x)$ be two functions with derivatives of $f'(x)$ and $g'(x)$, respectively.

$$(f(x) \times g(x))' = f'(x)g(x) + f(x)g'(x).$$

Theorem 5.2.3 Let $f(x)$ and $g(x)$ be two functions with derivatives of $f'(x)$ and $g'(x)$, respectively.

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

Theorem 5.2.4 For a composite function $f(g(x))$ or $(f \circ g)(x)$, the derivative will be

$$f'(g(x)) \times g'(x).$$

OR

If $y = f(u)$ and $u = g(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}.$$

7. Higher Order Differentiation:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2},\ f''(x),\ f'''(x),\ f^{(4)}(x),\ f^{(5)}(x),\ \cdots$$

5.3 Applications of Derivatives

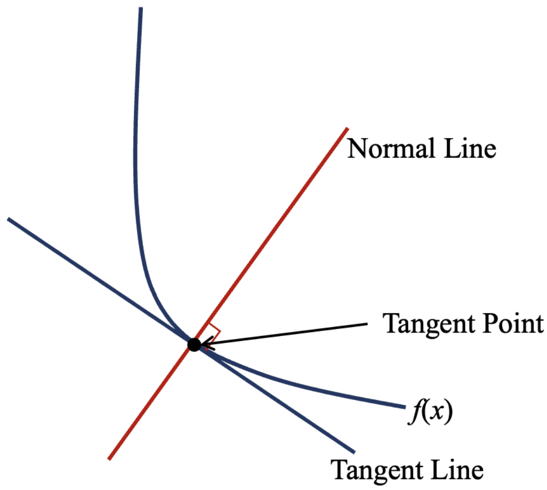
1. Equation of Tangent Line:
   Via the original functions, we could get the tangent point (x_0, y_0). Then, the expression of the tangent line is

   $$y - y_0 = m(x - x_0),$$

   where m is the derivative.

2. Normal and Tangent Lines:

   Definition 5.3.1 **Normal** is perpendicular to the tangent and passes through the same tangent point.



3. Increasing and Decreasing Function:

Definition 5.3.2 **Increasing Function**: As x is getting larger, y is getting larger.
i.e.,

$$\frac{\mathrm{d}y}{\mathrm{d}x} > 0.$$

Decreasing Function: As x is getting larger, y is getting smaller.
i.e.,

$$\frac{\mathrm{d}y}{\mathrm{d}x} < 0.$$

4. Local Extrema: $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ Stationary point
   Global extrema is the maximum and the minimum points of the entire function.
   $f''(x)$ is used to determine if the local extrema are maxima or minima.
   - Minima: $f''(x) > 0$ Concave up.
   - Maxima: $f''(x) < 0$ Concave down.
   - Point of Inflection (the point that is changing from concaving up to concaving down, or vice visa): $f''(x) = 0$
5. With local extrema, x-intercepts, y-intercepts, concavity, and asymptotes, draw approximate diagrams of a function.

5.4 Implicit Differentiation

1. When differentiating something with y, multiply $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the end.
2. $(y^2)' = 2y\frac{\mathrm{d}y}{\mathrm{d}x}$.

   Proof 5.4.1 If $u = y^2$, then

   $$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 2y \cdot \frac{\mathrm{d}y}{\mathrm{d}x}. \qquad \text{[Chain Rule]}$$

   Example 5.4.1 **Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the circle $x^2 + y^2 = 16$.**

   $$(x^2)' + (y^2)' = (16)' \;\Rightarrow\; 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
   $$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{2y} = -\frac{x}{y}.$$

   Example 5.4.2 **Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for $e^x + x\sin y = \cos 2y$.**

   $$\begin{aligned}(e^x)' + (x\sin y)' &= (\cos 2y)' \\ e^x + \left(\sin y + x\cos y\frac{\mathrm{d}y}{\mathrm{d}x}\right) &= -2\sin 2y\frac{\mathrm{d}y}{\mathrm{d}x} \\ (-x\cos y - 2\sin 2y)\frac{\mathrm{d}y}{\mathrm{d}x} &= e^x + \sin y \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{e^x + \sin y}{-x\cos y - 2\sin 2y}.\end{aligned}$$

3. Second Order Differentiation of Implicit functions*: Differentiate the first order differentiation.

Example 5.4.3 Find $\frac{d^2y}{dx^2}$ for the circle $x^2+y^2=16$. (From Ex. 4.1: $2x+2y\frac{dy}{dx}=0$, $\frac{dy}{dx}=-\frac{2x}{2y}=-\frac{x}{y}$.)

$$(2x)'+\left(2y\frac{dy}{dx}\right)'=(0)'\Rightarrow 2+\left((2y)'\frac{dy}{dx}+2y\left(\frac{dy}{dx}\right)'\right)=0\Rightarrow 2+2\left(\frac{dy}{dx}\right)^2+2y\frac{d^2y}{dx^2}=0$$

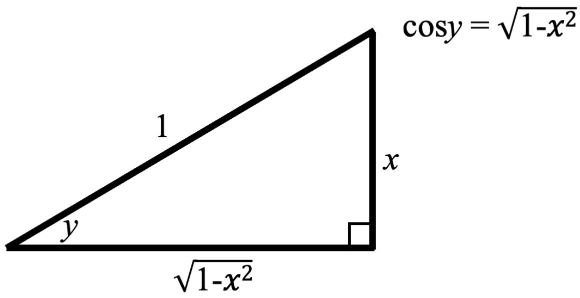
$$\frac{d^2y}{dx^2}=\frac{-2-2\left(\frac{dy}{dx}\right)^2}{2y}=\frac{-2-2\left(-\frac{x}{y}\right)^2}{2y}.$$

4. Derivative of Inverse Trigonometry Functions

Theorem 5.4.1

$$y=\arcsin x\ \Rightarrow\ \frac{dy}{dx}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-x^2}},\ \arcsin x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right](\cos y>0).$$

Proof 5.4.2 From $y=\arcsin x$, we get $\sin y=x$. This situation can be illustrated by the figure below:



$$\therefore (\sin y)'=(x)'\ \Rightarrow\ \cos y\frac{dy}{dx}=1$$

$$\therefore \frac{dy}{dx}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-x^2}}.$$

Theorem 5.4.2

$$y=\arccos x\ \Rightarrow\ \frac{dy}{dx}=-\frac{1}{\sin y}=-\frac{1}{\sqrt{1-x^2}},\ \arccos x\in[0,\pi]\,(\sin y>0).$$

$$y=\arctan x\ \Rightarrow\ \frac{dy}{dx}=\frac{1}{1+x^2}.$$

Proof 5.4.3 (Hint: Try to visualize a similar diagram as in proof 4.1.)
From $y=\arccos x$, we get $\cos y=x$.

$$\therefore (\cos y)'=(x)'\ \Rightarrow\ -\sin y\frac{dy}{dx}=1\ \Rightarrow\ \frac{dy}{dx}=-\frac{1}{\sin y}=-\frac{1}{\sqrt{1-x^2}}.$$

From $y = \arctan x$, we get $\tan y = x$.

$$\therefore (\tan y)' = (x)' \Rightarrow \sec^2 y \frac{\mathrm{d}y}{\mathrm{d}x} = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sec^2 y} = \cos^2 y = \left(\frac{1}{\sqrt{1+x^2}}\right) = \frac{1}{1+x^2}.$$

5.5 Related Rate of Change

1. When finding a rate of change of x, we are finding the $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Example 5.5.1 Area of circle is increasing at a rate of 10π per second. When the radius is 2, what is the rate of change of radius?
Known: $\frac{\mathrm{d}A}{\mathrm{d}t} = 10\pi$, $r = 2$. Find: $\frac{\mathrm{d}r}{\mathrm{d}t}$.

$$A = \pi r^2 \;\Rightarrow\; \frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi r \frac{\mathrm{d}r}{\mathrm{d}t} = 10\pi$$
$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{10\pi}{2\pi r} = \frac{5}{r}$$

$$\text{When } r = 2,\ \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{5}{2}.$$

Example 5.5.2 A spherical balloon is expanding at a rate of 60π per second. How fast is the surface area of the balloon expanding when the radius is 4?
Known: $\frac{\mathrm{d}V}{\mathrm{d}t} = 60\pi$, $r = 4$. Find $\frac{\mathrm{d}A}{\mathrm{d}t}$.

$$V = \frac{4}{3}\pi r^3 \;\Rightarrow\; \frac{\mathrm{d}V}{\mathrm{d}t} = 3 \cdot \frac{4}{3}\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t} = 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t}$$
$$\therefore 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t} = 60\pi \;\Rightarrow\; \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{60\pi}{4\pi r^2} = \frac{15}{r^2}$$
$$A = 4\pi r^2 \;\Rightarrow\; \frac{\mathrm{d}A}{\mathrm{d}t} = 8\pi r \frac{\mathrm{d}r}{\mathrm{d}t} = 8\pi r \cdot \frac{15}{r^2} = \frac{120\pi}{r}.$$

$$\text{When } r = 4,\ \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{120\pi}{4} = 30\pi.$$

2. Kinematics:

- Velocity, displacement, and acceleration are vector variables that have a value and a direction.
- Speed only has a value and no direction. It is a scalar variable. No sign should be reported in the answer.
- If s is the displacement, v is the velocity, a is the acceleration:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = v;\ \frac{\mathrm{d}v}{\mathrm{d}t} = a.$$

5.6 More Limits - L'Hopital's Rule

Theorem 5.6.1 When the limit is in the **indeterminant form** ($\frac{0}{0}$ or $\frac{\infty}{\infty}$),

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] \lim_{x\to a}\left[\frac{f'(x)}{g'(x)}\right],$$

where $f'(x)$ and $g'(x)$ are the first derivatives of $f(x)$ and $g(x)$, respectively.

Example 5.6.1

$$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{\sec^2 x}{1}=1$$

5.7 Indefinite Integration

1. Regard Integration as Anti-differentiation:

$$f'(x)=x \quad\Rightarrow\quad f(x)=\frac{1}{2}x^2+C,\ \text{where } C \text{ is a constant.}$$

$$f'(x)=x^2 \quad\Rightarrow\quad f(x)=\frac{1}{3}x^3+C,\ \text{where } C \text{ is a constant.}$$

$$f'(x)=x^n \quad\Rightarrow\quad f(x)=\frac{1}{n+1}x^{n+1}+C,\ \text{where } C \text{ is a constant.}$$

Definition 5.7.1 Anti-differentiation is also called **indefinite integration**. It is denoted by $\int\ \mathrm{d}x$.

e.g. $\int x^n\ \mathrm{d}x=\frac{1}{n+1}x^{n+1}+C$.

2. General Rules of Integration.

- $$\int x^n\ \mathrm{d}x=\frac{1}{n+1}x^{n+1}+C$$
- $$\int k\ \mathrm{d}x=kx+C$$
- $$\int kf(x)\ \mathrm{d}x=k\int f(x)\ \mathrm{d}x$$
- $$\int f(x)\pm g(x)\ \mathrm{d}x=\int f(x)\ \mathrm{d}x\pm\int g(x)\ \mathrm{d}x$$

3. $\int f'(x)\ \mathrm{d}x=f(x)+C$. Therefore, if we know the $f'(x)$ and a point on the $f(x)$, which is to determine the constant C, then we can deduce the original function $f(x)$.

4. More Rules of Integration:

Differentiation	Integration
$(e^x)'=e^x$	$\int e^x\ \mathrm{d}x=e^x+C$
$(\ln x)'=frac1x$	$\int \frac{1}{x}\ \mathrm{d}x=\ln x+C$
$(\sin x)'=\cos x$	$\int \cos x\ \mathrm{d}x=\sin x+C$
$(\cos x)'=-\sin x$	$\int \sin x\ \mathrm{d}x=-\cos x+C$
$(\tan x)'=\sec^2 x$	$\int \sec^2 x\ \mathrm{d}x=\tan x+C$
$(\cot x)'=-\csc^2 x$	$\int \csc^2 x\ \mathrm{d}x=-\cot x+C$
$(\sec x)'=\sec x\tan x$	$\int \sec x\tan x\ \mathrm{d}x=\sec x+C$
$(\csc x)'=-\csc x\cot x$	$\int \csc x\cot x\ \mathrm{d}x=-\csc x+C$

5. Anti-chain Rule in Integration: We must divide the chain rule factor.

Example 5.7.1

$$\int (ax+b)^n \, \mathrm{d}x = \frac{1}{a}\left(\frac{1}{n+1}(ax+b)^{n+1}\right)+C$$

$$\int e^{(ax+b)} \, \mathrm{d}x = \frac{1}{a}e^{ax+b}+C$$

$$\int \frac{1}{(ax+b)} \, \mathrm{d}x = \frac{1}{a}\ln(ax+b)+C$$

$$\int \sin(ax+b) \, \mathrm{d}x = -\frac{1}{a}\cos(ax+b)+C$$

$$\int \cos(ax+b) \, \mathrm{d}x = \frac{1}{a}\sin(ax+b)+C$$

6. Integration Techniques: by Substitution and by Parts:

Theorem 5.7.1 Whenever we have an integration like: $\int g'(x)\times f(g(x)) \, \mathrm{d}x$, we can always assume $u = g(x)$. Therefore, $\mathrm{d}u = g'(x)\cdot \mathrm{d}x$. $()u = g(x) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = g'(x))$:

$$\int g'(x)\times f(g(x)) \, \mathrm{d}x = \int f(u \, \mathrm{d}u).$$

Theorem 5.7.2 If $f(x)$ and $g(x)$ are two functions, and $f'(x)$ and $g'(x)$ are their derivatives, respectively, integration by parts can be written as following:

$$\int f(x)g'(x) \, \mathrm{d}x = f(x)g(x) - \int f'(x)g(x) \, \mathrm{d}x.$$

Example 5.7.2 Find $\int 2x(x^2+3)^5 \, \mathrm{d}x$.
Since $2x = (x^2+3)'$, we consider to use integration by substitution.
Assume $u = x^2+3$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = (x^2+3)' = 2x \Rightarrow \mathrm{d}u = 2x\cdot\mathrm{d}x$.

$$\begin{aligned}\therefore \int 2x(x^2+3)^5 \, \mathrm{d}x &= \int (x^2+3)^5\cdot(2x\cdot\mathrm{d}x)\\ &= \int u^5 \, \mathrm{d}u\\ &= \frac{1}{6}u^6+C\\ &= \frac{1}{6}(x^2+3)^6+C.\end{aligned}$$

7. Integration of Inverse Trigonometric Functions:

- $$\int \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x = \arcsin x + C$$
- $$\int \frac{1}{1+x^2} \, \mathrm{d}x = \arctan x + C$$
- $$\int \frac{1}{x\sqrt{x^2-1}} \, \mathrm{d}x = \mathrm{arcsec} x + C$$

Example 5.7.3 **Find** $\int \frac{\mathrm{d}x}{x^2+4x+5}$

$$\int \frac{\mathrm{d}x}{x^2+4x+5} = \frac{\mathrm{d}x}{(x+2)^2+1}$$

$$\text{Assume } u = x+2,\ \frac{\mathrm{d}u}{\mathrm{d}x} = 1 \ \Rightarrow\ \mathrm{d}u = \mathrm{d}x$$

$$\therefore \int \frac{\mathrm{d}x}{x^2+4x+5} = \frac{\mathrm{d}u}{u^2+1}$$
$$= \arctan u + C$$
$$= \arctan(x+2) + C.$$

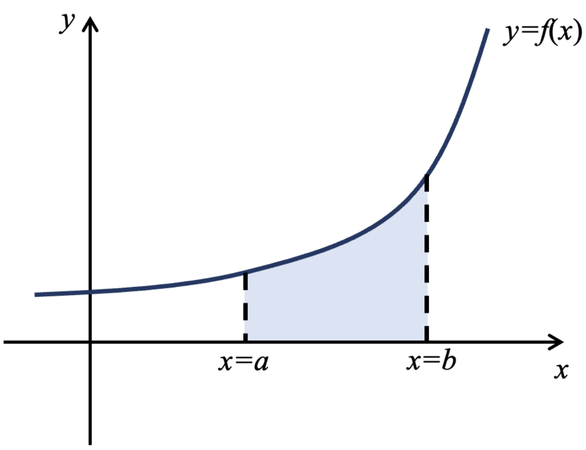
5.8 Approximating the Area Under a Curve

1. The definite integral is equal to the limit at infinity of the Riemann sum, and hence gives the exact area under the curve between $x = a$ and $x = b$. i.e.,

$$\lim_{n\to\infty} = \sum_{i}^{n} f(x_i)\Delta x_i = \int_a^b f(x)\ \mathrm{d}x,$$

where a is the lower limit and b is the upper limit.

2. If $f(x) \geq 0\ \forall x \in [a,b]$, then $\int_a^b f(x)\ \mathrm{d}x$ is defined as the shaded area:



This is known as the Riemann integral.

3. The Fundamental Theorem of Calculus:

Theorem 5.8.1 For a continuous function $f(x)$ with antiderivative $F(x)$:

$$\int_a^b f(x)\ \mathrm{d}x = F(b) - F(a).$$

This theorem explains the link between differential calculus and the definite integral.

4. Properties of Definite Integrals:

- $$\int_a^a f(x)\ \mathrm{d}x = 0$$

- $$\int_a^b d\ \mathrm{d}x = k(b-a),\ (k \text{ is a constant}).$$

- $$\int_b^a f(x)\ \mathrm{d}x = -\int_a^b f(x)\ \mathrm{d}x$$

- $$\int_a^b kf(x)\ \mathrm{d}x = k\int_a^b f(x)\ \mathrm{d}x$$

- $$\int_a^b f(x)\ \mathrm{d}x + \int_b^c f(x)\ \mathrm{d}x = \int_a^c f(x)\ \mathrm{d}x$$

- $$\int_a^b [f(x) \pm g(x)]\ \mathrm{d}x = \int_a^b f(x)\ \mathrm{d}x \pm \int_a^b g(x)\ \mathrm{d}x$$

5. When the function $f(x)$ is negative for $x \in [a,b]$, then the area bounded by the curve, the x-axis and the lines $x=a$ and $x=b$ is given by

$$\left|\int_a^b f(x)\ \mathrm{d}x\right|.$$

6. Finding Areas Between Two Functions:

   - Sketch: find the intersections and determine which function is above.
   - Integration.

5.9 Volumes of Revolution

1. The volume of a solid of revolution formed when $y=f(x)$, which is continuous in the interval $[a,b]$, is rotated 2π radians about the x-axis is given by

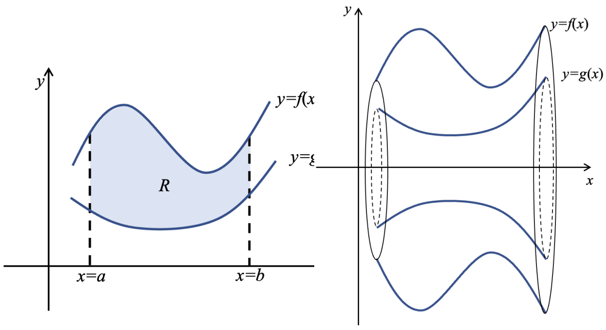
$$V = \pi\int_a^b y^2\ \mathrm{d}x.$$

2. The volume of a solid of revolution formed when $y=f(x)$, which is continuous in the interval $y=c$ to $y=d$, is rotated 2π radians about the y-axis is given by

$$V = \pi\int_c^d x^2\ \mathrm{d}y.$$

3. Consider a region R between two curves, $y=f(x)$ and $y=g(x)$, from $x=a$ to $x=b$, when $f(x) > g(x)$.

   - Rotating R about the x-axis generates a solid of revolution S. The criss-section of this



(a) (b)

shape looks like a washer whose area is given by:

$$A = \pi(R^2 - r^2) = \pi\left((f(x))^2 - (g(x))^2\right).$$

So the volume of S is given by:

$$\begin{aligned} V &= \int_a^b A(x)\,\mathrm{d}x \\ &= \int_a^b \left((f(x))^2 - (g(x))^2\right)\,\mathrm{d}x. \end{aligned}$$

- Rotating R about the y-axis in the interval $c \leq y \leq d$:

$$V = \pi\int_c^d \left((x_1)^2 - (x_2)^2\right)\,\mathrm{d}y,$$

where x_1 and x_2 are expression of x with respect to y of $f(x)$ and $g(x)$.

5.10 Differential Equation

1. Differential Equation:

Definition 5.10.1 A **differential equation** is an equation containing the derivatives of one or more dependent variables with respect to one or more independent variables. Equation that involves the derivatives of one or more functions.

E.g.

$$y' = 6x + 1 \qquad \text{Lagrange notation}$$

$$3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{dx} - y = 3 \qquad \text{Leibniz notation}$$

$$f'(x) = 6x + 1 \qquad \text{Function notation}$$

The independent variable is x, and the dependent variable is y. The solution to a differential equation is a function or a set of functions.

2. Two Types of Differential Equations:

Definition 5.10.2 **Ordinary Differential Equations (ODEs)**: deals with functions of a single variable and ordinary derivatives. **Partial Differential Equations (PDEs)**: deals with multi-variable equations and their partial derivatives (with more than one independent variables).

3. Order of Differential Equations:

   Definition 5.10.3 The **order of the differential equation** is the highest order derivative in the equation.

4. Linearity of ODEs:

   Theorem 5.10.1 A differential equation is said to be **linear** if:

   - All the terms with dependent variables are in first-order.
   - The coefficients of all the terms in the dependent variable and its derivatives depend only on the independent variable x.

5. Linear First-Order ODEs:

   Definition 5.10.4

   $$\frac{\mathrm{d}y}{\mathrm{d}x} + a(x)y = b(x), \text{ where } a(x) \text{ and } b(x) \text{ are functions of } x.$$

6. Solutions of ODEs:

   - The solution to an ODE is a function or a set of functions.
   - General solution to the differential equation:
     For a differential equation of order n, a solution is a function that satisfies the equation on some interval I. The function should have at least its first n derivatives on this interval I.
   - To find particular solutions, we need to initial conditions for the problem.
     (a) Initial Value Problem (IVP): where initial values are given to solve the differential equations depending on the order of the ODE.
     E.g. $y(0)$, $t(0)$, $(0,y)$.
     (b) Boundary Value Problem: where a certain boundary is given.
     E.g. (x,y).

7. Separable Differential Equations:

   Definition 5.10.5 A differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y)$ is **separable** if it can be expressed as a product of a function in x and a function in y:

   $$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) = g(x)h(y).$$

   - Particularly, if $h(x) \neq 0$, the variable can be separated to

   $$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y) \Rightarrow \frac{\mathrm{d}y}{h(y)} = g(x)\,\mathrm{d}x$$
   $$\int \frac{\mathrm{d}y}{h(y)} = \int g(x)\,\mathrm{d}x$$

- Solving differential equations using separation of variables:
  (a) Separate the variables such that everything involving y is on one side and everything involving x is on the other side.
  (b) Integrate both sides and combine the constant of integration on one side of the equation (normally the right side).

Example 5.10.1 Solve for y if $\frac{\mathrm{d}y}{\mathrm{d}x} = x(1+y)e^x$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= x(1+y)e^x \\
\frac{1}{1+y}\mathrm{d}y &= xe^x \mathrm{d}x \\
\int \frac{1}{1+y}\,\mathrm{d}y &= \int xe^x\,\mathrm{d}x \\
\left(= xe^x - \int e^x\,\mathrm{d}x = xe^x - e^x\right. & \left.\text{[Integration by Parts]}\right) \\
\ln|1+y| &= xe^x - e^x + C \\
1+y &= e^{xe^x - e^x + C} = e^{xe^x - e^x} \cdot e^C \\
y &= Ae^{xe^x - e^x} - 1 \quad (A = e^C).
\end{aligned}$$

8. The Standard Logistic Equation:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = kn(a-n); \quad a,k \in \mathbb{R}.$$

where t is the time during which a population grows,
n is the population after time t,
k is the relative growth, and
a is a constant.

9. Homogeneous Differential Equations:

Definition 5.10.6 Differential equations of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right)$, where $y = y(x)$, are known as **homogeneous differential equations**.

Theorem 5.10.2 Homogeneous differential equations can be solved by using the substitution $y = vx$, where v is a function of x. The substitution will always reduce the differential equation to a separable differential equation.

Proof 5.10.1 If $y = vx$, where v is a function of x, then:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}v}{\mathrm{d}x}x + v \qquad \text{[Product Rule]} \\
\because \frac{\mathrm{d}y}{\mathrm{d}x} &= f\left(\frac{y}{x}\right), \\
\therefore \frac{\mathrm{d}v}{\mathrm{d}x}x + v &= f\left(\frac{y}{x}\right) = f(v) \\
\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{f(v) - v}{x} \\
\Rightarrow \frac{1}{f(v) - v}\,\mathrm{d}v &= \frac{1}{x}\,\mathrm{d}x
\end{aligned}$$

Example 5.10.2 Solve for $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+2y}{x}$, given $y(3) = \frac{3}{2}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + 2\frac{y}{x} \;\rightarrow\; \text{homogeneous differential equation}$$

$$\text{Let } y = vx,\; \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}x}x + v \;\Rightarrow\; \frac{\mathrm{d}v}{\mathrm{d}x}x + v = 1 + 2\frac{y}{x} = 1 + 2v.$$

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{1+v}{x} \\
\frac{1}{1+v}\,\mathrm{d}v &= \frac{1}{x}\,\mathrm{d}x \\
\int \frac{1}{1+v}\,\mathrm{d}v &= \int \frac{1}{x}\,\mathrm{d}x \\
\ln|1+v| &= \ln|x| + C = ln|Ax| \\
1 + v = Ax \;&\Rightarrow \frac{y}{x} + 1 = Ax \\
y &= Ax^2 - x.
\end{aligned}$$

Substituting $y = \frac{3}{2}$, $x = 3$: $\frac{3}{2} = A(3)^2 - 3 \;\Rightarrow\; A = \frac{1}{2}$

$$\therefore y = \frac{1}{2}x^2 - x.$$

Example 5.10.3 Solve for $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y}{x}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{y}{x} \;\rightarrow\; \text{homogeneous differential equation}$$

$$\text{Assume } v = \frac{y}{x}:\; y = vx \;\Rightarrow\; \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}x}x + v.$$

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= 1+v = \frac{\mathrm{d}v}{\mathrm{d}x}x+v \\
\mathrm{d}v &= \frac{1}{x}\,\mathrm{d}x \\
\int \mathrm{d}v &= \int \frac{1}{x}\,\mathrm{d}x \\
v &= \ln|x|+C = \ln|Ax| \\
\frac{y}{x} &= \ln|Ax| \\
y &= x\ln|Ax|.
\end{aligned}
$$

10. Using the Integrating Factor $I(x)$:

Definition 5.10.7

$$I(x) = e^{\int P(x)\,\mathrm{d}x}$$

is the **integrating factor** for $\frac{\mathrm{d}y}{\mathrm{d}x}+P(x)y=Q(x)$, where P and Q are continuous functions of x on a given interval.

Theorem 5.10.3

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x}+P(x)y &= Q(x) \\
I(x)\frac{\mathrm{d}y}{\mathrm{d}x}+I(x)P(x)y &= I(x)Q(x) \text{ [Multiply both sides by } I(x)\text{]} \\
\left(\frac{\mathrm{d}}{\mathrm{d}x}(I(x)y) = I(x)\frac{\mathrm{d}y}{\mathrm{d}x}+I(x)P(x)y \text{ [Product Rule]}\right)
\end{aligned}
$$

$$\left[(I(x))' = (e^{\int P(x)\,\mathrm{d}x})' = e^{\int P(x)\,\mathrm{d}x}\cdot\left(\int P(x)\,\mathrm{d}x\right)' = e^{\int P(x)\,\mathrm{d}x}\cdot P(x) = I(x)P(x)\right]$$

$$
\begin{aligned}
\therefore \frac{\mathrm{d}}{\mathrm{d}x}(I(x)y) &= I(x)Q(x) \\
\int \frac{\mathrm{d}}{\mathrm{d}x}(I(x)y)\,\mathrm{d}x &= \int I(x)Q(x)\,\mathrm{d}x \\
I(x)y &= \int I(x)Q(x)\,\mathrm{d}x.
\end{aligned}
$$

Example 5.10.4 Solve $\frac{\mathrm{d}y}{\mathrm{d}x}+3x^2y=6x^2$.

$$\because P(x)=3x^2,\ Q(x)=6x^2,$$

$$\therefore I(x) = e^{\int P(x)\,\mathrm{d}x} = e^{\int 3x^2\,\mathrm{d}x} = e^{x^3}.$$

Multiply both sides by $I(x)$:

$$e^{x^3}\frac{\mathrm{d}y}{\mathrm{d}x}+e^{x^3}\cdot 3x^2y=e^{x^3}\cdot 6x^2$$
$$\therefore \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^3}y\right)=e^{x^3}\cdot 6x^2$$
$$\int \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^3}y\right)\,\mathrm{d}x=\int e^{x^3}\cdot 6x^2\,\mathrm{d}x$$
$$\left[\text{Let } x^3=u,\ \frac{\mathrm{d}u}{\mathrm{d}x}=2x^2,\ \mathrm{d}u=2x^2\,\mathrm{d}x \Rightarrow\ 2\int e^u\,\mathrm{d}u=2e^u+C=2e^{x^3}+C\right]$$
$$e^{x^3}y=2e^{x^3}+C$$
$$y=2+Ce^{-x^3}.$$

11. Euler's Method:

- For $y=f(x)$, $y_{n+1}=y_n+hf'(x_0)$, h is a constant.

$$y-y_n=f'(x_n)(x-x_n).$$

Example 5.10.5 $y=x^2,\ \frac{\mathrm{d}y}{\mathrm{d}x}=2x,\ h=0.1$

n	x_n	y_n	Actual
0	1	1	1
1	1.1	1.2	1.21
2	1.2	1.42	1.44
3	1.3	1.66	1.69
4	1.4	1.92	1.96
5	1.5	2.2	2.25

- The smaller the h, the more accurate the approximation.
- Consider a differential equation of the form $\frac{\mathrm{d}y}{\mathrm{d}x}=f(x,y)$, given an initial condition. The derivative at any point on the curve $(x_0,y(x_0))$ can be approximated using the gradient of the tangent to the curve at x_0:

$$y'(x_0)=\frac{y(x_0+h)-y(x_0)}{h}.$$

Rearranging the formula, we get:

$$y(x_0+h)=y(x_0)+hy'(x+0).$$

This is the **linearization** or **Euler's method** and becomes more accurate over small increments and as long as the function does not change too rapidly.

- If $\frac{\mathrm{d}y}{\mathrm{d}x}=f(x_n,y-n)$ and $x_{n+1}=x_n+h$, we have

$$y_{n+1}=y_n+hf(x_n,y_n).$$

5.11 Maclaurin Series

1. The Maclaurin Polynomial:

   Definition 5.11.1 If $f(x)$ has n derivatives at $x=0$, then $P(x)$, the **Maclaurin polynomial** of degree n for $f(x)$ centered at $x=0$, is the unique polynomial of degree n that satisfies:

   - $$f(0)=P(0);$$
   - $$f^{(n)}(0)=P^{(n)}(0);$$
   - $$a_1=\frac{f'(0)}{1!},\ a_2=\frac{f''(0)}{2!},\ a_3=\frac{f'''(0)}{3!},\ \cdots\ a_n=\frac{f^{(n)}(0)}{n!};$$
   - $$P_n(x)=f(0)++\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n=\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k.$$

2. Maclaurin polynomials approximate the behavior of functions around a certain interval. The more terms we take, the better the approximation.

3. The Maclaurin Series:

   Definition 5.11.2 If $f(x)$ has derivatives of all orders throughout an open interval I such that $0\in I$, then the **Maclaurin series** generated by f at $x=0$ is:

   $$P_n(x)=f(0)++\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\cdots=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n.$$

   A series converges when the sum of them is a constant (a limit can be found).

   Example 5.11.1 Find the Maclaurin series for $f(x)=\frac{1}{2+x}$.

$f(x)=(2+x)^{-1}$	$f(0)=2^{-1}=\frac{1}{2}$
$f'(x)=-(2+x)^{-2}$	$f'(0)=-2(2)^{-2}=-\frac{1}{4}$
$f''(x)=2(2+x)^{-3}$	$f''(0)=2(2)^{-3}=2\times\frac{1}{8}$
$f'''(x)=-6(2+x)^{-4}$	$f'''(0)=-6(2)^{-4}=-6\times\frac{1}{16}$
$f^{(4)}(x)=24(2+x)^{-5}$	$f^{(4)}(0)=24(2)^{-5}=24\times\frac{1}{32}$

   $$\begin{aligned}P(x)&=\frac{1}{2}+\frac{-\frac{1}{4}}{1!}x+\frac{2\times\frac{1}{8}}{2!}x^2+\frac{-6\times\frac{1}{16}}{3!}x^3+\frac{24(2)^{-5}=24\times\frac{1}{32}}{4!}x^4+\cdots\\&=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1}(-x)^n.\end{aligned}$$

4. The Binomial series is the Maclaurin expansion for $f(x)=(1+x)^p$:

   $$(1+x)^p=\sum_{n=0}^{p}\binom{p}{n}x^n,\ 1\le n\le p,\ \binom{p}{n}=\frac{p!}{n!(p-n)!}=\frac{p(p-1)(p-2)\cdots(p-(n-1))}{n!}.$$

Example 5.11.2 Use the Binomial series to find the Maclaurin series for $f(x) = \frac{1}{(x+2)^2}$.

$$\begin{aligned}
f(x) &= (1+x)^{-2} \\
\because \binom{-2}{n} &= \frac{-2(-2-1)(-2-2)\cdots(-2-(n-1))}{n!} \\
&= (-1)^n \frac{2(3)(4)\cdots(n+1)}{n!} = (-1)^n(n+1) \\
\therefore P(x) &= \sum_{n=0}^{\infty} (-1)^n(n+1)x^n \\
&= 1 - 2x + 3x^2 - 4x^3 + \cdots + (-1)^n(n+1)x^n + \cdots
\end{aligned}$$

Example 5.11.3 Use the Binomial series to find the Maclaurin series for $f(x) = \frac{1}{\sqrt{2-x}}$.

$$f(x) = (2-x)^{-\frac{1}{2}} = (2)^{-\frac{1}{2}} \left(1 - \frac{x}{2}\right)^{-\frac{1}{2}} = \frac{\sqrt{2}}{2}\left(1 - \frac{x}{2}\right)^{-\frac{1}{2}}$$

$$\therefore P(x) = \sum_{n=0}^{\infty} \frac{\sqrt{2}}{2} \binom{-\frac{1}{2}}{n} \left(1 - \frac{x}{2}\right)^{-\frac{1}{2}} = \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(1 - \frac{x}{2}\right)^{-\frac{1}{2}}.$$

5. Applications of Maclaurin Series:

- Approximation of sin, cos, tan,...

  Example 5.11.4 Approximate $\sin 3^\circ$ using the first four terms of Maclaurin series.

$$3^\circ = \frac{\pi}{60}, \text{ For } \sin x,\ P(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\therefore P\left(\frac{\pi}{60}\right) = x - \frac{\left(\frac{\pi}{60}\right)^3}{3!} + \frac{\left(\frac{\pi}{60}\right)^5}{5!} - \frac{\left(\frac{\pi}{60}\right)^7}{7!} + \cdots \approx 0.052336 \text{ (6 d.p.).}$$

- More Complicated Functions

  Example 5.11.5 Find the Maclaurin series of $f(x) = e^{x^2}$.
  Let $u = x^2, f(x) = e^u$:

$$P(x) = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{u^n}{n!} = \sum_{n=0}^{\infty} \frac{\left(x^2\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.$$

  Example 5.11.6 Find the Maclaurin series of $f(x) = \ln\left(\frac{1+x}{1-x}\right)$.

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

$$\begin{aligned}
\therefore P(x) &= -\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots-\left(x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots\right)\\
&= 2(x+\frac{x^3}{3}+\cdots)\\
&= 2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}.
\end{aligned}$$

Example 5.11.7 Find the Maclaurin series of $f(x)=\frac{x}{(1+x)^2}$.

$$\begin{aligned}
f(x) &= x(1+x)^{-2}\\
&= x\sum_{n=0}^{\infty}\binom{-2}{n}x^n\\
&= \sum_{n=0}^{\infty}(-1)^n(n+1)x^{n+1}.
\end{aligned}$$

- Evaluate Limits

Example 5.11.8 Find $\lim\limits_{x\to 0}\frac{1-e^{x^2}}{1-\cos x}$.

$$e^{x^2} = 1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\cdots$$

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$$

$$\begin{aligned}
\therefore \lim_{x\to 0}\frac{1-e^{x^2}}{1-\cos x} &= \lim_{x\to 0}\frac{1-\left(1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\cdots\right)}{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)}\\
&= \lim_{x\to 0}\frac{-x^2-\frac{x^4}{2!}-\frac{x^6}{3!}-\cdots}{\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\cdots}\\
&= \lim_{x\to 0}\frac{-x^2}{\frac{x^2}{2!}}\\
&= -2.
\end{aligned}$$

[Consider only the smallest power of x, as higher powers will go to zero much quicker.]

- Solve Differential Equations

Example 5.11.9 Use the first six terms of a Maclaurin series to approximate the solution of $y'=y^2-x$ on an open interval centered at $x=0$ if $y(0)=1$.

	$y(0)=1$
$y'=y^2-x$	$y'(0)=1$
$y''=2yy'-1$	$y''(0)=2-1=1$
$y'''=2yy''+2(y')^2$	$y'''(0)=2+2=4$
$y^{(4)}=2yy'''+6y'y''$	$y^{(4)}(0)=14$
$y^{(5)}=2yy^{(4)}+8y'y'''+6(y'')^2$	$y^{(5)}(0)=66$

$$\therefore P(x) = 1 + x + \frac{1}{2}x^2 + \frac{4}{3!}x^3 + \frac{14}{4!}x^4 + \frac{66}{5!}x^5 + \cdots$$

- Binomial Theorem

Theorem 5.11.1 Function $f(x) = (1+x)^p,\ p \in \mathbb{R}$ is equal to its Binomial series using the initial condition $y(0) = 1$.

Proof 5.11.1

$$f(x) = 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}x^n + \cdots$$

$$\therefore f'(x) = p + p(p-1)x + \frac{p(p-1)(p-2)}{2!}x^2 + \cdots$$

$$xf'(x) = px + p(p-1)x^2 + \frac{p(p-1)(p-2)}{2!}x^3 + \cdots$$

$$\begin{aligned} \therefore f'(x) + xf'(x) &= p + [p(p-1) + p]x + \left[\frac{p(p-1)}{2!}p(p-1) + \right]x^2 + \cdots \\ &= p + p^2x + \frac{p^2(p-1)}{2!}x^2 + \cdots \\ &= p(1 + px + \frac{p(p-1)}{2!}x^2) + \cdots \\ &= pf(x) \end{aligned}$$

$$\therefore f'(x) + xf'(x) = pf(x) \ \Rightarrow\ (1+x)f'(x) = pf(x)$$

$$f'(x) - \frac{p}{1+x}f(x) = 0 \ \Rightarrow\ P(x) = -\frac{p}{1+x},\ Q(x) = 0$$

$$\therefore I(x) = e^{\int -\frac{p}{1+x}\,\mathrm{d}x} = A(1+x)^{-p}$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}x}\left(A(1+x)^{-p}f(x)\right) = 0 \cdot I(x)$$

$$\int \frac{\mathrm{d}}{\mathrm{d}x}\left(A(1+x)^{-p}f(x)\right)\,\mathrm{d}x = \int 0\,\mathrm{d}x$$

$$A(1+x)^{-p}f(x) = C$$

$$f(x) = C \cdot A(1+x)^{x-p} = B(1+x)^{x-p} \qquad [\text{Let } B = C \cdot A.]$$

$$\text{Let } f(0) = 1:\ B = 1$$

$$\therefore f(x) = (1+x)^p.$$