Emory University
MATH 211 - Advanced Calculus (Multivariable)
Learning Notes

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Contents

1 Vectors and Geometry of Space

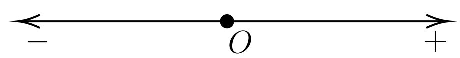
1.1 Three Dimensional Coordinate System

Definition 1.1.1 (Coordinate System). A **coordinate system** is a system that uses coordinate of a point to uniquely determine the position of the point in the space or plane.

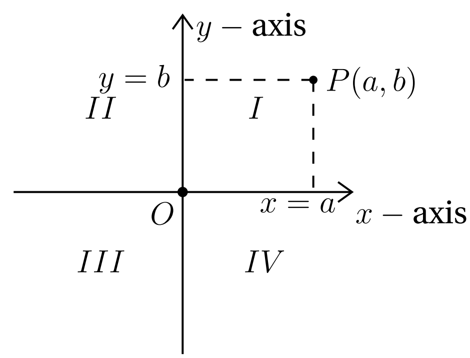
The Cartesian coordinate system is defined in different dimensions.

Definition 1.1.2 (One Dimensional Cartesian System). One Dimensional Cartesian System is a straight line with a fixed point as the origin and positive and negative directions.

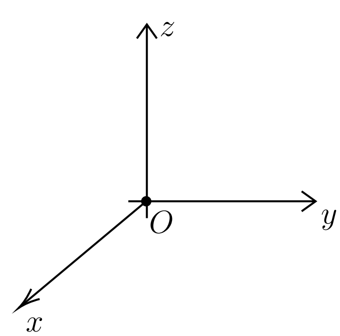
Remark. The one dimensional cartesian system is the number line:



Any point in the one dimensional Cartesian system corresponds to a number $\in \mathbb{R}$ and any number $\in \mathbb{R}$ has a location on the line. The two dimensional Cartesian system is the regular coordinate system.

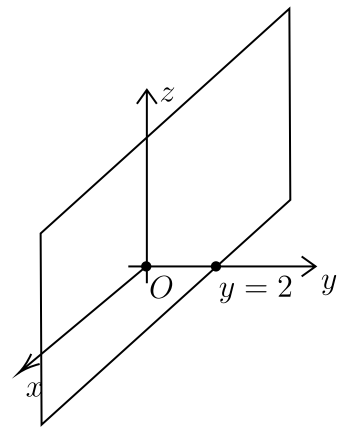


The three dimensional Cartesian system includes three perpendicular axes.

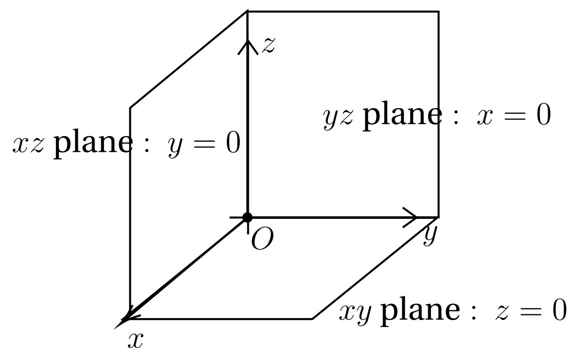


Definition 1.1.3 (Octant). A **Octant** is one of the eight divisions of the three dimensional coordinate system.

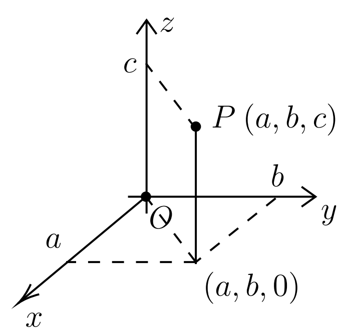
Definition 1.1.4 (Hyperplane). The hyperplane of $y = 2$ is given as below:



Specially:



Definition 1.1.5 (Points in the Three Dimensional System). $P(a, b, c)$ indicates the intersection of the three hyperplanes: $x = a$, $y = b$, and $z = c$.



For spaces in the higher dimension, we understand them via the Cartesian product.

Definition 1.1.6 (Cartesian Product).

$$\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{R} \forall i = 1, \cdots, n\}$$

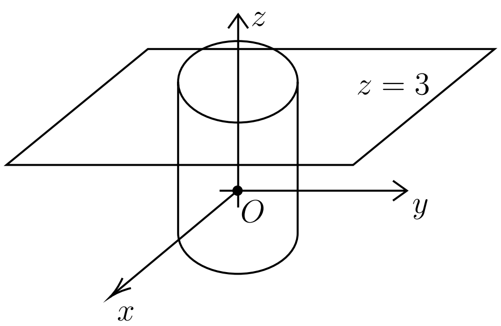
is the set of all n-tuples of real numbers and is denoted by $\mathbb{R}^n$.

Example 1.1.1. $(3, 4, 5) \in \mathbb{R}^3$ is 3 dimensional. $(3, 4, 5, 6) \in \mathbb{R}^4$ is 4 dimensional.

Example 1.1.2. Which point(s) (x, y, z) satisfies the equations

$$x^2 + y^2 = 1 \quad \text{and} \quad x = 3?$$

Answer.



Those points form a circle in the hyperplane of $z = 3$ centered at the point $(0, 0, 3)$ with a radius of 1.

□

Theorem 1.1.1 (Distance Formula in Three Dimension). For given points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, the distance between them is denoted by $|P_1P_2|$ and is defined by

$$|P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

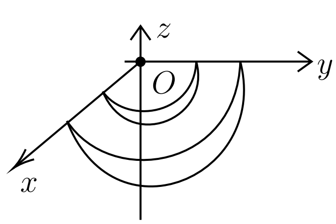
Theorem 1.1.2 (Equation of a Sphere). An equation of a sphere with a center of (a, b, c) and a radius of r is defined as

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

Example 1.1.3. What is the region in $\mathbb{R}^3$ represented by the inequalities

$$1 \leq x^2 + y^2 + z^2 \leq 4 \quad \text{and} \quad z \leq 0?$$

Answer.

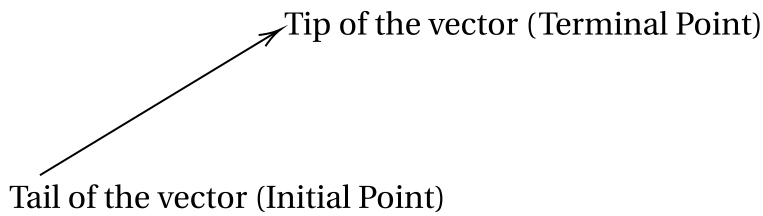


The region is the difference between the half spheres (the lower half of the sphere) centered at $(0, 0, 0)$ with a radius of 1 and 2.

□

1.2 Vectors

Definition 1.2.1 (Vectors). Vectors are used to indicate a quantity that has both magnitude and direction.



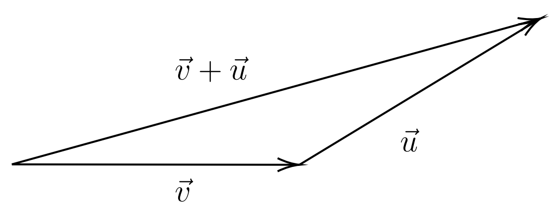
1. Vectors are denoted as $\vec{v}$.
2. Magnitude

   Definition 1.2.2 (Magnitude). A vector is a line segment, of which the **magnitude** of vector denoted by $|\vec{v}|$ is the length of it and the arrow points the direction of the vector.

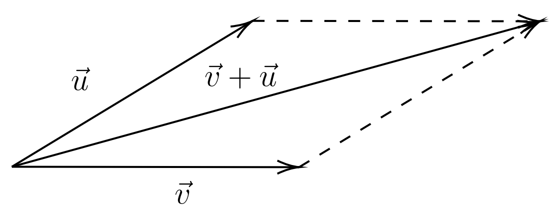
Vectors are operated in a different way:

1. Addition of Vectors:

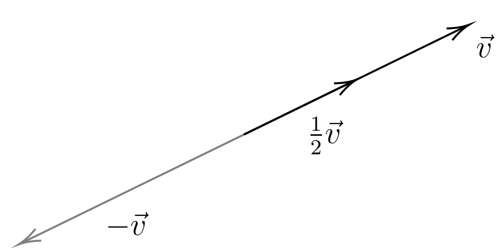
   (a) The triangle law:

   (b) The parallelogram law:

2. Scalar Multiplications:

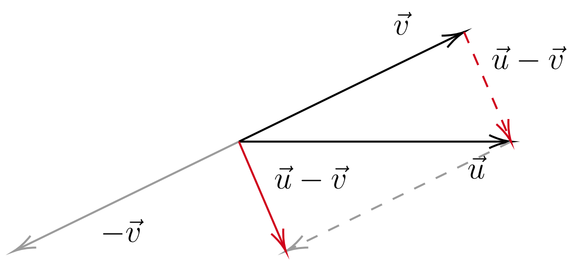


Definition 1.2.3 (Scalar Multiplication). If $c \in \mathbb{R}$ and $\vec{v}$ is a vector, then $c\vec{v}$ is in the same direction of $\vec{v}$ if $c > 0$ and in the opposite direction if $c < 0$.

Theorem 1.2.1. The magnitude of $c\vec{v}$:

$$|c\vec{v}| = c|\vec{v}|.$$

3. Differences of Vectors:



The difference of vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ is denoted by $\vec{\mathbf{u}} - \vec{\mathbf{v}}$ and is defined by

$$\vec{\mathbf{u}} - \vec{\mathbf{v}} = \vec{\mathbf{u}} + (-\vec{\mathbf{v}})$$

4. Properties of vectors:
Suppose $\vec{\mathbf{a}},\ \vec{\mathbf{b}},\ \vec{\mathbf{c}}$ are vectors in V_n and c and d are scalars (*Those properties can be proven geometrically*):

   (a) $\vec{\mathbf{a}} + \vec{\mathbf{b}} = \vec{\mathbf{b}} + \vec{\mathbf{a}}$

   (b) $\vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = (\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}}$

   (c) $\vec{\mathbf{a}} + 0 = \vec{\mathbf{a}}$

   (d) $\vec{\mathbf{a}} + (-\vec{\mathbf{a}}) = 0$

   (e) $c(\vec{\mathbf{a}} + \vec{\mathbf{b}}) = c\vec{\mathbf{a}} + c\vec{\mathbf{b}}$

   (f) $(c + d)\vec{\mathbf{a}} = c\vec{\mathbf{a}} + d\vec{\mathbf{a}}$

   (g) $(cd)\vec{\mathbf{a}} = c(d\vec{\mathbf{a}})$

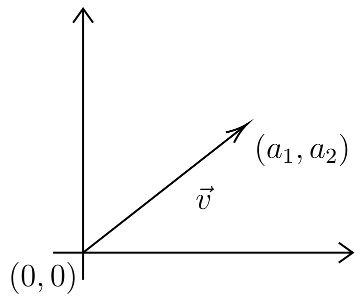
   (h) $1 \cdot \vec{\mathbf{a}} = \vec{\mathbf{a}}$

We can link the coordinate system and vectors together:

1. **Definition 1.2.4 (Components of Vectors).** We will denote vector $\vec{\mathbf{v}}$ as

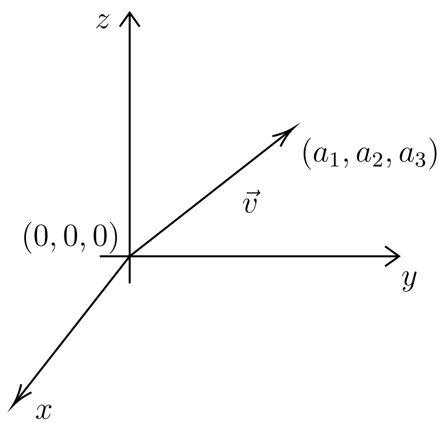
$$\vec{\mathbf{v}} = \langle a_1, a_2 \rangle,$$

where a_1 and a_2 are called the **components** of $\vec{\mathbf{v}}$.



2. In the three dimension:

$$\vec{\mathbf{v}} = \langle a_1, a_2, a_3 \rangle$$



3. **Definition 1.2.5.** If $A(x_1, y_1, z_1)$ as the tail of vector $\vec{\mathbf{v}}$ and $B(x_2, y_2, z_2)$ as the tip of vector $\vec{\mathbf{v}}$, then

$$\overrightarrow{AB} = \langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \rangle$$

$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

4. **Theorem 1.2.2.** If $\vec{\mathbf{v}} = \langle a, b, c \rangle$ and $\vec{\mathbf{u}} = \langle a', b', c' \rangle$, then

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \langle a' + a,\ b' + b,\ c' + c \rangle$$

$$\vec{\mathbf{u}} - \vec{\mathbf{v}} = \langle a' - a,\ b' - b,\ c' - c \rangle$$

$$\alpha\vec{\mathbf{u}} = \langle \alpha a',\ \alpha b',\ \alpha c' \rangle, \text{ where } \alpha \text{ is a scalar.}$$

Definition 1.2.6 (Standard Basis Vectors). In 2-D, $\hat{\mathbf{i}} = \langle 1, 0 \rangle$ and $\hat{\mathbf{j}} = \langle 0, 1 \rangle$; and in 3-D, $\hat{\mathbf{i}} = \langle 1, 0, 0 \rangle$, $\hat{\mathbf{j}} = \langle 0, 1, 0 \rangle$, and $\hat{\mathbf{k}} = \langle 0, 0, 1 \rangle$ are called the **standard basis vectors**.

Remark. Any vectors in 2D and 3D can be written as

$$\vec{\mathbf{v}} = \langle a, b, c \rangle = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}.$$

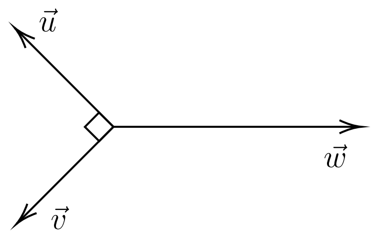
Definition 1.2.7 (Unit Vector). A **unit vector**is a vector of magnitude of 1.

Example 1.2.1.

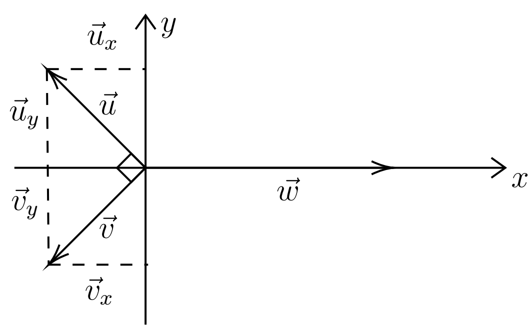
$$|\hat{\mathbf{i}}| = |\hat{\mathbf{j}}| = |\hat{\mathbf{k}}| = 1 \text{ are unit vectors.}$$

Theorem 1.2.3. To find a unit vector in the direction of any vector $\vec{\mathbf{v}}$, we use $\dfrac{1}{|\vec{\mathbf{v}}|}\vec{\mathbf{v}}$. The length of vector $\dfrac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$ is 1 and its direction is the same as $\vec{\mathbf{v}}$.

Example 1.2.2. If the vectors in the figure satisfy $|\vec{\mathbf{u}}| = |\vec{\mathbf{v}}| = 1$, and $\vec{\mathbf{u}} + \vec{\mathbf{v}} + \vec{\mathbf{w}} = 0$, find $|\vec{\mathbf{w}}|$.



Answer.
Decompose the vectors:



We then have

$$\cos 45^\circ = \frac{|\vec{\mathbf{u}}_x|}{\vec{\mathbf{u}}} \implies |\vec{\mathbf{u}}_x| = |\vec{\mathbf{u}}| \cos 45^\circ;$$

$$\sin 45^\circ = \frac{|\vec{\mathbf{u}}_y|}{\vec{\mathbf{u}}} \implies |\vec{\mathbf{u}}_y| = |\vec{\mathbf{u}}| \sin 45^\circ;$$

$$\begin{aligned} \therefore \vec{\mathbf{u}} = \langle |\vec{\mathbf{u}}_x|,\ |\vec{\mathbf{u}}_y| \rangle &= -|\vec{\mathbf{u}}_x|\hat{\mathbf{i}} + |\vec{\mathbf{u}}_y|\hat{\mathbf{j}} \\ &= -\frac{\sqrt{2}}{2}|\vec{\mathbf{u}}|\hat{\mathbf{i}} + \frac{\sqrt{2}}{2}\hat{\mathbf{j}} \\ &= \frac{\sqrt{2}}{2}|\vec{\mathbf{u}}|(-\hat{\mathbf{i}} + \hat{\mathbf{j}}) \end{aligned}$$

Similarly,

$$\vec{\mathbf{v}} = \frac{\sqrt{2}}{2}|\vec{\mathbf{v}}|(-\hat{\mathbf{i}} - \hat{\mathbf{j}}).$$

We know $\vec{\mathbf{u}} + \vec{\mathbf{v}} + \vec{\mathbf{w}} = 0$:

$$\therefore \vec{\mathbf{w}} + \frac{\sqrt{2}}{2}|\vec{\mathbf{u}}|(-\hat{\mathbf{i}} + \hat{\mathbf{j}}) + \frac{\sqrt{2}}{2}|\vec{\mathbf{v}}|(-\hat{\mathbf{i}} - \hat{\mathbf{j}}) = 0$$

We know $|\vec{\mathbf{u}}| = |\vec{\mathbf{v}}| = 1$:

$$\begin{aligned} \therefore \vec{\mathbf{w}} + \frac{\sqrt{2}}{2}(-\hat{\mathbf{i}} + \hat{\mathbf{j}}) + \frac{\sqrt{2}}{2}(-\hat{\mathbf{i}} - \hat{\mathbf{j}}) &= 0 \\ \vec{\mathbf{w}} + \frac{\sqrt{2}}{2}(-\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{i}} - \hat{\mathbf{j}}) &= 0 \\ \vec{\mathbf{w}} &= \sqrt{2}\hat{\mathbf{i}} \end{aligned}$$

$$\therefore \vec{\mathbf{w}} = \langle \sqrt{2}, 0 \rangle \implies |\vec{\mathbf{w}}| = \sqrt{2}.$$

□

1.3 Dot Product

Definition 1.3.1 (Dot Product). If $\vec{\mathbf{u}} = \langle x_1, y_1, z_1 \rangle$ and $\vec{\mathbf{v}} = \langle x_2, y_2, z_2 \rangle$, then the dot product of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ is defined as

$$\begin{aligned} \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= \langle x_1, y_1, z_1 \rangle \cdot \langle x_2, y_2, z_2 \rangle \\ &= x_1x_2 + y_1y_1 + z_1z_2 \end{aligned}$$

Remark. The dot product of two vectors returns a scalar.

Example 1.3.1. Let $\vec{\mathbf{u}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ and $\vec{\mathbf{v}} = 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$. Find $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$.

Answer.

$$\begin{aligned} \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= \langle 1, 2, -3 \rangle \cdot \langle 0, 2, -1 \rangle \\ &= (1)(0) + (2)(2) + (-3)(-1) = 7. \end{aligned}$$

□

Properties of the dot product:

1. $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{b}} \cdot \vec{\mathbf{a}}$
2. $\vec{\mathbf{a}} \cdot (\vec{\mathbf{v}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}}\vec{\mathbf{b}} + \vec{\mathbf{a}}\vec{\mathbf{c}}$
3. $m(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}) = (m\vec{\mathbf{a}}) \cdot \vec{\mathbf{b}} = \vec{\mathbf{a}} \cdot (m\vec{\mathbf{b}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})m$
4. $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$
   $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0$

Theorem 1.3.1.

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{u}} = |\vec{\mathbf{u}}|^2.$$

Theorem 1.3.2. If θ is the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, then

$$\boxed{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{u}}| \cdot |\vec{\mathbf{v}}| \cos\theta}.$$

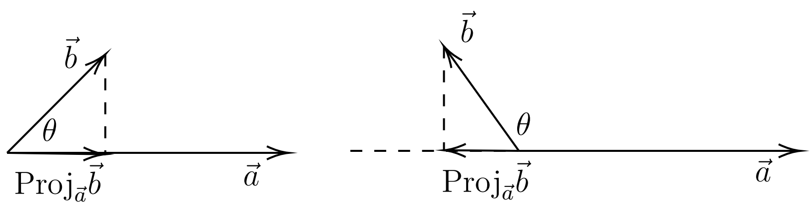
Extension.

$$\cos\theta = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{|\vec{\mathbf{u}}||\vec{\mathbf{v}}|}$$

Extension.

$$\theta = 90^\circ \Longleftrightarrow \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0.$$

Definition 1.3.2 (Projections). We use $\mathrm{Proj}_{\vec{\mathbf{a}}}\vec{\mathbf{b}}$ to denote the **projection** of $\vec{\mathbf{b}}$ on $\vec{\mathbf{a}}$.



From the diagrams,

$$\cos\theta = \frac{|\mathrm{Proj}_{\vec{\mathbf{a}}}\vec{\mathbf{b}}|}{|\vec{\mathbf{b}}|} \Longrightarrow |\mathrm{Proj}_{\vec{\mathbf{a}}}\vec{\mathbf{b}} = \boxed{|\vec{\mathbf{b}}| \cos\theta}.$$

We know that

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}| \cos\theta$$

$$\therefore \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}|} = \boxed{|\vec{\mathbf{b}}| \cos\theta}$$

$$\therefore |\mathrm{Proj}_{\vec{\mathbf{a}}}\vec{\mathbf{b}}| = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}|}, \text{ which is a scalar.}$$

$|\mathrm{Proj}_{\vec{\mathbf{a}}}\vec{\mathbf{b}}|$ is called the **scalar projection** of $\vec{\mathbf{b}}$ on $\vec{\mathbf{a}}$.

$$\mathrm{Proj}_{\vec{\mathbf{a}}}\vec{\mathbf{b}} = |\mathrm{Proj}_{\vec{\mathbf{a}}}\vec{\mathbf{b}}| \cdot \underbrace{\frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|}}_{\text{unit vector}} = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}|} \cdot \frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|} = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}|^2} \cdot \vec{\mathbf{a}}$$

$\mathrm{Proj}_{\vec{\mathbf{a}}}\vec{\mathbf{b}}$ is called **projection** of $\vec{\mathbf{b}}$ on $\vec{\mathbf{a}}$ and is a vector.

Example 1.3.2. Find the scalar projection and vector projection of vector $\vec{\mathbf{u}} = \langle 1, 1, 2\rangle$ onto $\vec{\mathbf{v}} = \langle -2, 3, 1\rangle$.

Answer.

$$\text{Proj}_{\vec{\mathbf{v}}}\vec{\mathbf{u}} = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{|\vec{\mathbf{v}}|^2} \cdot \vec{\mathbf{v}}\,; \quad |\text{Proj}_{\vec{\mathbf{v}}}\vec{\mathbf{u}}| = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$$

We need $|\vec{\mathbf{v}}| = \sqrt{4+9+1} = \sqrt{14}$ and $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = (1)(-2) + (1)(3) + (2)(1) = 3$

$$\therefore |\text{Proj}_{\vec{\mathbf{v}}}\vec{\mathbf{u}}| = \frac{3}{\sqrt{14}}$$

$$\text{Proj}_{\vec{\mathbf{v}}}\vec{\mathbf{u}} = \frac{3}{14} \cdot \vec{\mathbf{v}} = \frac{3}{14} \cdot \langle -2, 3, 1 \rangle = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle.$$

□

1.4 Cross Product

Definition 1.4.1 (Cross Product). The **cross product** of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ is denoted by $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ and is a vector that is perpendicular to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. If $\vec{\mathbf{u}} = \langle x_1, y_1, z_1 \rangle$ and $\vec{\mathbf{v}} = \langle x_2, y_2, z_2 \rangle$, then

$$\begin{aligned} \vec{\mathbf{u}} \times \vec{\mathbf{v}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = y_1 z_2 \hat{\mathbf{i}} + x_2 z_1 \hat{\mathbf{j}} + x_1 y_2 \hat{\mathbf{k}} - x_2 y_1 \hat{\mathbf{k}} - y_2 z_1 \hat{\mathbf{i}} - x_1 z_2 \hat{\mathbf{j}} \\ &= (y_1 z_2 - y_2 z_1)\hat{\mathbf{i}} + (z_1 x_2 - z_2 x_1)\hat{\mathbf{j}} + (x_1 y_2 - x_2 y_1)\hat{\mathbf{k}} \end{aligned}$$

Example 1.4.1. Prove $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is perpendicular to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

Proof.

$$\begin{aligned} \vec{\mathbf{u}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) &= \langle x_1, y_1, z_1 \rangle \cdot \langle y_1 z_2 - y_2 z_1,\ z_1 x_2 - z_2 x_1,\ x_1 y_2 - x_2 y_1 \rangle \\ &= x_1 y_1 z_2 - x_z y_2 z_1 + x_2 y_1 z_1 - x_1 y_1 z_2 + x_1 y_2 z_1 - x_2 y_1 z_1 = 0 \\ &\therefore \vec{\mathbf{u}} \times \vec{\mathbf{v}} \perp \vec{\mathbf{u}} \end{aligned}$$

Similarly, $\vec{\mathbf{v}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = 0 \Longrightarrow \vec{\mathbf{u}} \times \vec{\mathbf{v}} \perp \vec{\mathbf{v}}$.

■

Theorem 1.4.1. If θ is the angle between vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, then

$$|\vec{\mathbf{u}} \times \vec{\mathbf{v}}| = |\vec{\mathbf{u}}||\vec{\mathbf{v}}| \sin\theta.$$

Proof.

$$\begin{aligned}
|\vec{\mathbf{u}} \times \vec{\mathbf{v}}|^2 &= (y_1z_2 - y_2z_1)^2 + (z_1x_2 - z_2x_1)^2 + (x_1y_2 - x_2y_1)^2 \\
&= (x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2) - (x_1x_2 + y_1y_2 + z_1z_2)^2 \\
&= |\vec{\mathbf{u}}|^2|\vec{\mathbf{v}}|^2 - (\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})^2 \\
&= |\vec{\mathbf{u}}|^2|\vec{\mathbf{v}}|^2 - |\vec{\mathbf{u}}|^2|\vec{\mathbf{v}}|^2\cos^2\theta \\
&= |\vec{\mathbf{u}}|^2|\vec{\mathbf{v}}|^2(1 - \cos^2\theta) \\
&= |\vec{\mathbf{u}}|^2|\vec{\mathbf{v}}|^2\sin^2\theta
\end{aligned}$$

$$\therefore |\vec{\mathbf{u}} \times \vec{\mathbf{v}}| = |\vec{\mathbf{u}}||\vec{\mathbf{v}}||\sin\theta|.$$

■

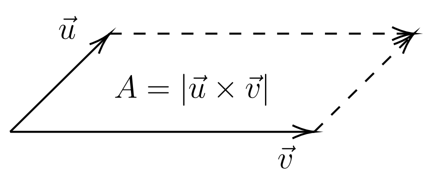
Definition 1.4.2 (Parallel). If two vectors, $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, are parallel to each other,

$$\vec{\mathbf{u}} = c\vec{\mathbf{v}},$$

where c is a scalar.

Theorem 1.4.2. For two vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = 0$ *iff* $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are parallel to each other.

Theorem 1.4.3. The length of the cross product, $|\vec{\mathbf{u}} \times \vec{\mathbf{v}}|$, is the area of the parallelogram determined by the vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.



Theorem 1.4.4.

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}; \quad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}; \quad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}; \quad \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}; \quad \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$$

Properties of cross product ($\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$ are vectors, and c is a scalar):

1. $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = -\vec{\mathbf{b}} \times \vec{\mathbf{a}}$
2. $(c\vec{\mathbf{a}}) \times \vec{\mathbf{b}} = c(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) = \vec{\mathbf{a}} \times (c\vec{\mathbf{b}})$
3. $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{a}} \times \vec{\mathbf{c}}$
4. $(\vec{\mathbf{a}} + \vec{\mathbf{b}}) \times \vec{\mathbf{c}} = \vec{\mathbf{a}} \times \vec{\mathbf{c}} + \vec{\mathbf{b}} \times \vec{\mathbf{c}}$
5. $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \cdot \vec{\mathbf{c}}$

6. $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\vec{\mathbf{b}} - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}}$

Definition 1.4.3 (Triple Product). The **scalar triple product** is defined by

$$\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}}).$$

Theorem 1.4.5. $|\vec{\mathrm{a}} \cdot (\vec{\mathrm{b}} \times \vec{\mathrm{c}})|$ denotes the volume of the parallelepiped determined by $\vec{\mathrm{a}}$, $\vec{\mathrm{b}}$, and $\vec{\mathrm{c}}$.

Proof.

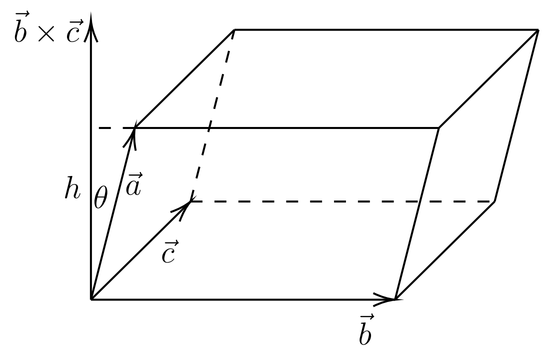
The area of the base is given by

$$A = |\vec{\mathbf{b}} \times \vec{\mathbf{c}}|$$

To find the volume, we need to know the height h:

$$h = |\vec{\mathbf{a}}||\cos\theta|$$

$$\therefore V = Ah = |\vec{\mathbf{b}} \times \vec{\mathbf{c}}||\vec{\mathbf{a}}||\cos\theta| = \vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) \qquad \left[\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}|\cos\theta\right]$$

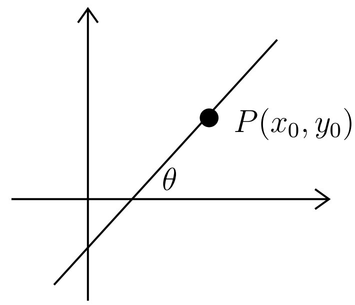


■

1.5 Equations of Lines and Planes

Theorem 1.5.1 (Equation of Lines in 2D). If we have a point $P(x_0, y_0)$ and a direction (slope/θ/another point on the line), we have the equation of the line:

$$\text{Given} \begin{cases} \text{slope} = m \\ P(x_0, y_0) \end{cases} \implies \text{The equation of the line: } y - y_0 = m(x - x_0).$$

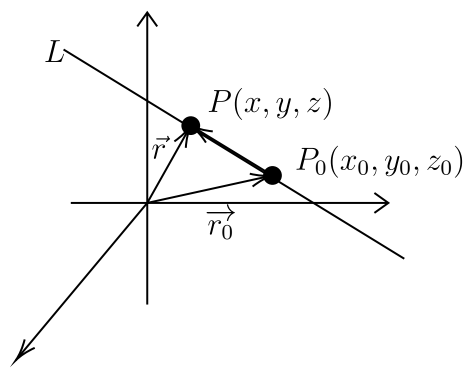


Definition 1.5.1 (Directional Vector). If $\vec{\mathbf{v}}$ is a directional vector of line L,

$$\vec{\mathbf{a}} = t\vec{\mathbf{v}},$$

where $\vec{\mathbf{a}}$ is any vector determined by two points on the line.

Definition 1.5.2 (Vector Equations of Lines in 3D). Let $\overrightarrow{P_0P} = \vec{\mathbf{a}} \Longrightarrow \vec{\mathbf{a}} = \langle x - x_0,\ y - y_0,\ z - z_0 \rangle$



From the diagram, we also have

$$\vec{\mathbf{r}}_0 + \vec{\mathbf{a}} = \vec{\mathbf{r}}.$$

As $\vec{\mathbf{a}} = t\vec{\mathbf{v}}$,

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}},$$

which is the **vector equation** of line L.

Theorem 1.5.2. If L is a line with point $P(x_0, y_0, z_0)$ on it and paralleled to a direction vector $\vec{\mathbf{v}} = \langle a, b, c \rangle$, we have

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle,$$

where t is a parameter and the equation is called the **vector equation** of line L.

Extension (Parametric Equation of *L*). From $\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$, we have

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

This system of equations is called the **parametric equation** of L.

Extension (Symmetric Equation of L). From the parametric equation of L, we can derive t:

$$\begin{cases} x = x_0 + ta & \implies \quad t = \frac{x-x_0}{a} \\ y = y_0 + tb & \implies \quad t = \frac{y-y_0}{b} \\ z = z_0 + tc & \implies \quad t = \frac{z-z_0}{c} \end{cases}$$

As t should be equal:

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c},$$

which is called the **symmetric equation** of the line with point $P(x_0, y_0, z_0)$ and a directional vector $\vec{\mathbf{v}} = \langle a, b, c\rangle$.

Remark (Three Forms of Equation of a Line). For line L in 3D, $P_0(x_0, y_0, z_0)$ is on L and $\vec{\mathbf{v}} = \langle a, b, c\rangle$ is a directional vector of L.

1. The vector form:

$$\langle x, y, z\rangle = \langle x_0 + ta,\ y_0 + tb,\ z_0 + tc\rangle$$

2. The parametric form:

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

3. The symmetric form:

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

Example 1.5.1. Find the parametric and symmetric equations of the line L passing through the points $(-8, 1, 4)$ and $(3, -2, 4)$.

Answer.

Let's set P_0 to be $(-8, 1, 4)$ and P_1 to be $(3, -2, 4)$. So we can find the directional vector

$$\vec{\mathbf{v}} = \overrightarrow{P_0P_1} = \langle 3-(-8),\ -2-1,\ 4-4\rangle = \langle 11, -3, 0\rangle.$$

$\therefore$ The parametric equation of L:

$$\begin{cases} x = -8 + 11t \\ y = 1 - 3t \\ z = 4 + (0)t \end{cases},$$

and the symmetric equation of L is

$$\frac{x+8}{11} = \frac{y-1}{-3}, \quad z = 4.$$

□

Relationships of two lines in 3D:

1. Parallel: directional vectors of the two lines are parallel to each other.
2. Intersect: the two lines share one common point
3. Skewed: the two lines are neither parallel nor intersecting.

Example 1.5.2. Let

$$L_1 : \frac{x-2}{1} = \frac{y-3}{-2} = \frac{z-1}{-3} \quad \text{and} \quad L_2 : \frac{x-3}{1} = \frac{y+4}{3} = \frac{z-2}{-7}.$$

Find the relationship between L_1 and L_2.

Answer.

$$\vec{\mathbf{v}}_1 = \langle 1, -2, -3 \rangle; \quad \vec{\mathbf{v}}_2 = \langle 1, 3, -7 \rangle$$

Because $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ are not parallel to each other, L_1 and L_2 are not parallel to each other.
$\therefore L_1$ and L_2 can only be intersecting or skewed.
To further discuss the relationship between L_1 and L_2, form parametric equations:

$$L_1 : \begin{cases} x = 2 + t \\ y = 3 - 2t \\ z = 1 - 3t \end{cases} \qquad L_2 : \begin{cases} x = 3 + s \\ y = -4 + 3s \\ z = 2 - 7s \end{cases}$$

If we can find a set of solutions t and s that satisfy the following system of equations, the two lines have point in common and thus is intersecting:

$$\begin{cases} 2 + t = 3 + s \\ 3 - 2t = -4 + 3s \\ 1 - 3t = 2 - 7s \end{cases} \implies \begin{cases} t - s = 1 & \text{①} \\ 2t + 3s = 7 & \text{②} \\ 3t - 7s = -1 & \text{③} \end{cases}$$

From ①:

$$t = s + 1 \qquad \text{④}$$

Substitute ② with ④:

$$\begin{aligned} &2(s+1)+3s=7 \\ &2s+2+3s=7 \quad \Rightarrow \quad 4s=5 \quad \Rightarrow \quad s=1 \\ &\therefore t=s+1=1+1=2 \end{aligned}$$

Substitute $s=1$ and $t=2$ to ③:

$$\text{LHS} = 2(3)-7(1)=6-7=-1=\text{RHS}.$$

Hence, $\begin{cases} t=2 \\ s=1 \end{cases}$ satisfy all three equations. Substitute $t=2$ to L_1:

$$x=2+2=4, \quad y=3-2(2)=-1, \quad z=1-3(2)=-5.$$

$\therefore$ The two lines intersect at $(4,-1,-5)$.

□

Theorem 1.5.3 (Line Segment that Connects $\vec{\mathbf{r}}_0$ and $\vec{\mathbf{r}}_r$).

$$\vec{\mathbf{r}}(t)=(1-t)\vec{\mathbf{r}}_0+t\vec{\mathbf{r}}_1, \qquad 1\le t\le 1.$$

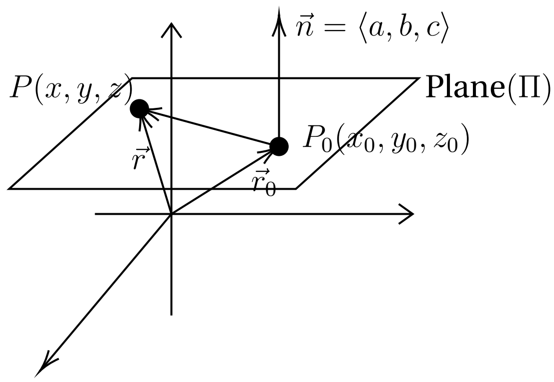
The vector equation gives a line segment the joins the tip of $\vec{\mathbf{r}}_0$ to the tip of $\vec{\mathbf{r}}_1$.

Definition 1.5.3 (Normal Vector). A normal vector is the vector perpendicular to the plane and is often denoted as $\vec{\mathrm{n}}$.

Theorem 1.5.4 (Vector Equation of a Plane). As $\vec{\mathrm{n}}\perp\Pi$, $\vec{\mathrm{n}}\perp\overrightarrow{P_0P}$

$$\begin{aligned} \overrightarrow{P_0P} &= \vec{\mathbf{r}}-\vec{\mathbf{r}}_0 \\ \therefore \vec{\mathrm{n}}\cdot(\vec{\mathbf{r}}-\vec{\mathbf{r}}_0) &= 0 \\ \vec{\mathrm{n}}\cdot\vec{\mathbf{r}}-\vec{\mathrm{n}}\cdot\vec{\mathbf{r}}_0 &= 0 \;\Rightarrow\; \vec{\mathrm{n}}\cdot\vec{\mathbf{r}}=\vec{\mathrm{n}}\cdot\vec{\mathbf{r}}_0, \end{aligned}$$

which is called the **vector equation** of a plane.



Extension (Scalar Equation of a Plane). From $\vec{\mathbf{n}} \cdot (\vec{\mathbf{r}} - \vec{\mathbf{r}}_0) = 0$: As $\vec{\mathbf{n}} = \langle a, b, c\rangle$ and $\vec{\mathbf{r}} - \vec{\mathbf{r}}_0 = \langle x - x_0,\ y - y_0,\ z - z_0\rangle$, we have

$$\langle a, b, c\rangle \cdot \langle x - x_0,\ y - y_0,\ z - z_0\rangle = 0;$$

$$\therefore a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

which is the **scalar equation** of plane Π with point $P_0(x_0, y_0, z_0)$ on it and a normal vector $\vec{\mathbf{n}} = \langle a, b, c\rangle$.

Extension (Linear Equation of a Plane). From $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$:

$$ax + by + cz - (ax_0 + by_0 + cz_0) = 0$$

Take $d = -(ax_0 + by_0 + cz_0)$:

$$ax + by + cz + d = 0,$$

which is called the **linear equation** of plane Π with point $P_0(x_0, y_0, z_0)$ on it and a normal vector $\vec{\mathbf{n}} = \langle a, b, c\rangle$.

Remark (Equations of a Plane). If point $P_0(x_0, y_0, z_0)$ is on the plane Π and a normal vector of Π is $\vec{\mathbf{n}} = \langle a, b, c\rangle$:

1. The vector equation:
$$\vec{\mathbf{n}} \cdot \vec{\mathbf{r}} = \vec{\mathbf{n}} \cdot \vec{\mathbf{r}}_0$$

2. The scalar equation:
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

3. The linear equation:
$$ax + by + cz + d = 0,$$
where $d = -(ax_0 + by_0 + cz_0) = -\langle a, b, c\rangle \cdot \langle x_0, y_0, z_0\rangle$

Example 1.5.3. Find an equation of the plane crossing through the points $P(1,3,2)$, $Q(3,-1,6)$, and $R(5,2,0)$.

Answer.

Find the normal vector using the following equation:

$$\vec{\mathbf{n}} = \overrightarrow{PQ} \times \overrightarrow{PR}$$

$$\overrightarrow{PQ} = \langle 3-1, -1-3, 6-2\rangle = \langle 2,-4,4\rangle$$
$$\overrightarrow{PR} = \langle 5-1, 2-3, 0-2\rangle = \langle 4,-1,-2\rangle$$

$$\therefore \vec{\mathbf{n}} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\hat{\mathbf{i}} + 20\hat{\mathbf{j}} + 14\hat{\mathbf{k}}.$$

$$\therefore \vec{\mathbf{n}} = \langle 12,20,14\rangle, \qquad P(1,3,2)$$

$$\therefore d = -\langle 12,20,14\rangle \cdot \langle 1,3,2\rangle = -(12+60+28) = -100.$$

$$\therefore \text{Linear Equation of } \Pi: \ 12x+20y+14z-100=0 \quad \Longrightarrow \quad 6x+10y+7z-50=0.$$

□

Theorem 1.5.5 (Relationship Between Two Planes). If $\vec{\mathbf{n}}_1$ is a normal vector of plane Π_1, and $\vec{\mathbf{n}}_2$ is a normal vector of plane Π_2 , then the angle between the two planes is given by

$$\theta = \cos^{-1}\left(\frac{\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2}{|\vec{\mathbf{n}}_1||\vec{\mathbf{n}}_2|}\right).$$

i.e., the angle between the planes is the angle between the normal vectors.

Theorem 1.5.6 (Distance from a Point to a Plane). Distance of the point $P(x_1, y_1, z_1)$ from the plane $ax+by+cz+d=0$:

$$D = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}} \tag{1}$$

OR

$$D = \frac{\vec{\mathbf{b}} \cdot \vec{\mathbf{n}}}{|\vec{\mathbf{n}}|}, \tag{2}$$

where $\vec{\mathbf{n}}$ is the normal vector.

Example 1.5.4. Find the distance between the parallel planes:

$$\Pi_1: \ 10x+2y-2z=5 \quad \text{and} \quad \Pi_2: \ 5x+y-z=1.$$

Answer.

Assume point $P(x_1, y_1, z_1)$ is on plane Π_1:

$$10x_1 + 2y_1 - 2z_1 = 5$$
$$\therefore\ 5x_1 + y_1 - z_1 = \frac{5}{2}$$

Applying formula 1: $\vec{\mathbf{n}} = \langle a, b, c\rangle = \langle 5, 1, -1\rangle,\ d = -1$:

$$\therefore\ D = \frac{|5x_1 + y_1 - z_1 + d|}{\sqrt{a^2+b^2+c^2}} = \frac{|\frac{5}{2} - 1|}{\sqrt{26+1+1}} = \frac{3/2}{\sqrt{27}} = \frac{3}{2\sqrt{27}} \left(= \frac{\sqrt{3}}{6}\right).$$

□

Extension. Find the distance between two parallel planes:

$$\Pi_1 :\ ax + by + cz + d = 0 \quad \text{and} \quad \Pi_2 :\ ax + by + cz + d' = 0.$$

Let point $P(x_1, y_1, z_1)$ on Π_1:

$$ax_1 + by_1 + cz_1 + d = 0$$

Apply formula 1:

$$D = \frac{|ax_1 + by_1 + cz_1 + d'|}{\sqrt{a^2+b^2+c^2}} = \frac{-d + d'}{\sqrt{a^2+b^2+c^2}}.$$

1.6 Cylinders and Quadric Surfaces

Definition 1.6.1 (Cylinders). A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.

Definition 1.6.2 (Quadric Surfaces). A **quadric surface** is the graph of a second-degree equation in three variables x, y, and z. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gz + Hy + Iz + J = 0,$$

where $A, B, C, \cdots, J$ are constants, but by translation and rotation it can be brought into one of the standard forms:

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0.$$

Remark. Graphs of Quadric Surfaces (Refer to Page 877 of the Book):

1. Ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

All traces are ellipses.
If $a = b = c$, the ellipsoid is a sphere.

2. Elliptic Paraboloid:
$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$
Horizontal traces are ellipses. Vertical traces are parabolas.
The variable raised to the first power indicates the axis of the paraboloid.

3. Hyperbolic Paraboloid:
$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$
Horizontal traces are hyperbolas. Vertical traces are parabolas.

4. Cone:
$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$
Horizontal traces are ellipses.
Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.

5. Hyperboloid of One Sheet:
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$
Horizontal traces are ellipses. Vertical traces are hyperbolas.
The axis of symmetry corresponds to the variable whose coefficient is negative.

6. Hyperboloid of Two Sheets:
$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$
Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas.
The two minus sign indicate two sheets.

2 Vector Functions

2.1 Vector Functions and Space Curves

Definition 2.1.1 (Component Functions). $f(t),\ g(t),\ h(t)$ are real valued function and are called **component functions** of $\vec{\mathbf{r}}(t)$. We write

$$\vec{\mathbf{r}}(t) = \langle f(t),\ g(t),\ h(t)\rangle = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}.$$

Definition 2.1.2 (Limit of Vector Functions). To find the limit of a vector function, we check its component functions. That is

$$\lim_{t\to a}\vec{\mathbf{r}}(t) = \left\langle \lim_{t\to a} f(t),\ \lim_{t\to a} g(t),\ \lim_{t\to a} h(t)\right\rangle$$

Definition 2.1.3 (Continuity of Vector Functions). A vector function $\vec{\mathbf{r}}(t)$ is continuous if

$$\lim_{t\to a}\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}(a).$$

Example 2.1.1. 1. Find the domain of

$$\vec{\mathbf{r}}(t) = \left\langle \ln(t+1),\ \frac{t}{\sqrt{9-t^2}},\ 2^t\right\rangle$$

Answer.

- Domain of $\ln(t+1)$: $D_1 L t+1>0,\ t>-1$
- Domain of $\dfrac{t}{\sqrt{9-t^2}}$ $:\ D_2:\ 9-t^2>0,\ -3<t<3$
- Domain of $2^t:\ D_3:\ \mathbb{R}$

  Find the intersection of domains of component functions:

$$D_1\cap D_2\cap D_3:\ -1<t<3\ (t\in(-1,3))$$

□

2. Find $\lim_{t\to 0}\vec{\mathbf{r}}(t)$.

   Answer.

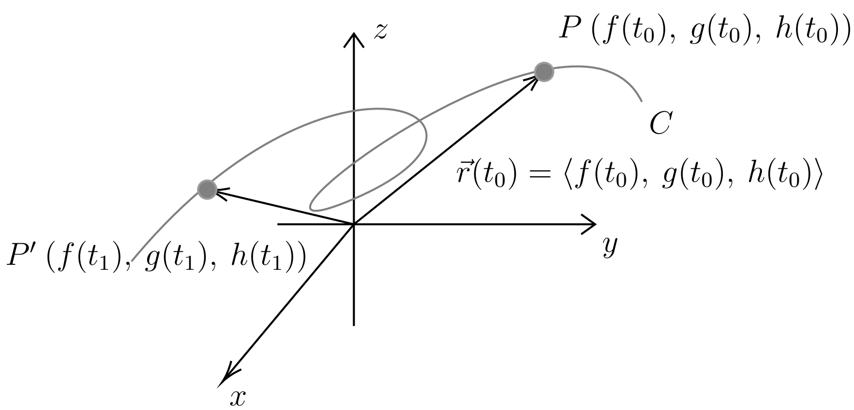
$$\begin{aligned}\lim_{t\to 0}\vec{\mathbf{r}}(t) &= \left\langle \lim_{t\to 0}\ln(t+1),\ \lim_{t\to 0}\frac{t}{\sqrt{9-t^2}},\ \lim_{t\to 0}2^t\right\rangle \\ &= \left\langle \ln(1),\ \frac{0}{\sqrt{9}},\ 2^0\right\rangle \\ &= \langle 0,0,1\rangle = \hat{\mathbf{k}}\end{aligned}$$

□

Example 2.1.2.

$$\begin{aligned}&\lim_{t\to 1}\left(\frac{t^2-t}{t-1}\hat{\mathbf{i}}+\sin\pi t\hat{\mathbf{j}}+\cos 2\pi t\hat{\mathbf{k}}\right)\\ =&\lim_{t\to 1}\left(\frac{t(t-1)}{t-1}\hat{\mathbf{i}}+\sin\pi t\hat{\mathbf{j}}+\cos 2\pi t\hat{\mathbf{k}}\right)\\ =&\lim_{t\to 1}t\hat{\mathbf{i}}+\lim_{t\to 1}\sin\pi t\hat{\mathbf{j}}+\lim_{t\to 1}\cos 2\pi t\hat{\mathbf{k}}\\ =&\hat{\mathbf{i}}+\sin\pi\hat{\mathbf{j}}+\cos 2\pi\hat{\mathbf{k}}\\ =&\hat{\mathbf{i}}+\hat{\mathbf{k}}\end{aligned}$$

Definition 2.1.4 (Graphs of Vector Functions). For a vector function $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$, the graph of it, curve C, is defined by the moving tip of the vectors yielded from the vector function.



Definition 2.1.5 (Space Curve). If f, g, h, are continuous real-valued functions on an interval I, then the set C of all points (x, y, z) in space s.t.

$$x = f(t) \quad y = g(t) \quad z = h(t), \quad \text{where} t \in I$$

is called a **space curve**.

Definition 2.1.6 (Parametric Equation). The system of equations $\begin{cases} x = f(t) \\ y = g(y) \\ z = h(t) \end{cases}$ is called a **parametric equation** of C and t is called the **parameter**.

2.2 Derivative and Intergral of Vector Functions

Limits, continuity, derivative, and integrals of vector functions follow rules similar to those of scalar functions.

Definition 2.2.1 (Derivative of Vector Functions).

$$\frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t} = \lim_{h \to 0} = \frac{\vec{\mathbf{r}}(t+h) - \vec{\mathbf{r}}(t)}{h},$$

$\dfrac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t}$ or $\vec{\mathbf{r}}'(t)$ is the derivative of $\vec{\mathbf{r}}(t)$ is the limit on the right hand side exists.

Extension. If $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$, then

$$\vec{\mathbf{r}}'(t) = f'(t)\hat{\mathbf{i}} + g'(t)\hat{\mathbf{j}} + h'(t)\hat{\mathbf{k}}.$$

Remark (Higher Order Derivatives). Higher order derivatives $\dfrac{\mathrm{d}^{(n)}\vec{\mathbf{r}}}{\mathrm{d}t^{(n)}}$ can be defined similarly.

Theorem 2.2.1 (Graphic Interpretation of Derivative). When $h \to 0$, the vector

$$\frac{\vec{\mathbf{r}}(t+h) - \vec{\mathbf{r}}(t)}{h}$$

becomes $\vec{\mathbf{r}}(t)$ and therefore, $\vec{\mathbf{r}}'(t)$ approaches to a vector that lies on the tangent line. $\vec{\mathbf{r}}'(t)$ is called the **tangent vector**, and

$$\vec{T} = \frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|}$$

is called the **unit tangent vector**.

Example 2.2.1. Find parametric equations of the tangent line to the vector function $\vec{\mathbf{r}}(t) = \langle 2\cos t,\ \sin t,\ t\rangle$ at point $\left(0, 1, \dfrac{\pi}{2}\right)$.

Answer.

When $t = \dfrac{\pi}{2}$, $2\cos\dfrac{\pi}{2} = 0,\ \sin\dfrac{\pi}{2} = 1.$

$\therefore \left(0, 1, \dfrac{\pi}{2}\right)$ is on the space curve of $\vec{\mathbf{r}}(t)$.

Find

$$\begin{aligned}\vec{\mathbf{r}}'(t) &= \langle (2\cos t)',\ (\sin t)',\ t' \rangle \\ &= \langle -2\sin t,\ \cos t, 1 \rangle\end{aligned}$$

When $t = \dfrac{\pi}{2}$,

$$\vec{\mathbf{r}}'\left(\frac{\pi}{2}\right) = \left\langle -2\sin\left(\frac{\pi}{2}\right),\ \cos\left(\frac{\pi}{2}\right), 1 \right\rangle = \langle -2, 0, 1 \rangle$$

$\therefore$ $\vec{\mathbf{d}}$ of tangent line $= \langle -2, 0, 1 \rangle$

$$\therefore \text{Line: } \left\langle 0, 1, \frac{\pi}{2} \right\rangle + \langle -2, 0, 1 \rangle t = \left\langle -2t, 1, \frac{\pi}{2} + t \right\rangle$$

□

Example 2.2.2. If $\vec{\mathbf{r}}(t) = (t^3 + 2t)\hat{\mathbf{i}} - 3e^{-2t}\hat{\mathbf{j}} + 2\sin 5t\hat{\mathbf{k}}$. Find $\dfrac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t}$, $\left|\dfrac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t}\right|$, $\dfrac{\mathrm{d}^2\vec{\mathbf{r}}}{\mathrm{d}t^2}$, $\left|\dfrac{\mathrm{d}^2\vec{\mathbf{r}}}{\mathrm{d}t^2}\right|$.

Answer.

$$\begin{aligned}\frac{\mathrm{d}\vec{\mathbf{r}}}{\mathrm{d}t} &= \langle 3t^2 + 2,\ 6e^{-2t},\ 10\cos 5t \rangle \\ \frac{\mathrm{d}^2\vec{\mathbf{r}}}{\mathrm{d}t^2} &= \langle 6t,\ -12e^{-2t},\ -50\sin 5t \rangle\end{aligned}$$

When $t = 0$:

$$\vec{\mathbf{r}}'(0) = \langle 2,\ 6,\ 10 \rangle; \qquad \vec{\mathbf{r}}''(0) = \langle 0,\ -12,\ 0 \rangle$$

$$\therefore |\vec{\mathbf{r}}'(0)| = \sqrt{4 + 36 + 100} = \sqrt{140}(= 2\sqrt{70}); \quad |\vec{\mathbf{r}}''(0)| = \sqrt{144} = 12.$$

□

Theorem 2.2.2 (Properties of Differentiation).

$$\frac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{r}}_1(t) + \vec{\mathbf{r}}_2(t)] = \frac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{r}}_1(t)] + \frac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{r}}_2(t)]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\alpha\vec{\mathbf{r}}(t)] = \alpha\frac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{r}}(t)]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[f(t)\vec{\mathbf{r}}(t)] = f'(t)\vec{\mathbf{r}}(t) + f(t)\vec{\mathbf{r}}'(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{r}}_1(t) \cdot \vec{\mathbf{r}}_2(t)] = \vec{\mathbf{r}}_1'(t) \cdot \vec{\mathbf{r}}_2(t) + \vec{\mathbf{r}}_1(t) \cdot \vec{\mathbf{r}}_2'(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{r}}_1(t) \times \vec{\mathbf{r}}_2(t)] = \vec{\mathbf{r}}_1'(t) \times \vec{\mathbf{r}}_2(t) + \vec{\mathbf{r}}_1(t) \times \vec{\mathbf{r}}_2'(t)$$

Example 2.2.3. Show that if a curve lies on a sphere with center at the origin, then $\vec{\mathbf{r}}'(t)$ is perpendicular to $\vec{\mathbf{r}}(t)$ for any t.

Answer.

Let $\vec{\mathbf{r}}(t)$ lies on a sphere, with center at the origin, and radius $R = c$:

$$\therefore \vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle \quad \text{and} \quad x^2(t) + y^2(t) + z^2(t) = c^2$$

$$x^2(t) + y^2(t) + z^2(t) = |\vec{\mathbf{r}}(t)|^2 = \vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}(t)$$

$$\therefore \vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}(t) = c^2$$

Take derivative of the both sides of the euqation

$$\frac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}(t)] = \frac{\mathrm{d}}{\mathrm{d}t}(c^2)$$

$$\therefore \vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{r}}(t) + \vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}'(t) = 0 \implies 2\vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{r}}(t) = 0$$

$$\therefore \vec{\mathbf{r}}'(t) \cdot \vec{\mathbf{r}}(t) = 0 \implies \vec{\mathbf{r}}'(t) \perp \vec{\mathbf{r}}(t).$$

□

Definition 2.2.2 (Definite Integral of a Vector Function). The definite integral of a continuous vector function $\vec{\mathbf{r}}(t)$ can be defined as

$$\int_a^b \vec{\mathbf{r}}(t)\mathrm{d}t = \int_a^b f(t)\mathrm{d}t\hat{\mathbf{i}} + \int_a^b g(t)\mathrm{d}t\hat{\mathbf{j}} + \int_a^b h(t)\mathrm{d}t\hat{\mathbf{k}},$$

if $\vec{\mathbf{r}}(t) = \left\langle f(t), g(t), h(t)\right\rangle$.

Example 2.2.4.

$$\begin{aligned}\int_0^1 \left(\frac{1}{t+1}\hat{\mathbf{i}} + \frac{1}{t^2+1}\hat{\mathbf{j}} + \frac{t}{t^2+1}\hat{\mathbf{k}}\right)\mathrm{d}t &= \int_0^1 \frac{1}{t+1}\mathrm{d}t\hat{\mathbf{i}} + \int_0^1 \frac{1}{t^2+1}\mathrm{d}t\hat{\mathbf{j}} + \int_0^1 \frac{t}{t^2+1}\mathrm{d}t\hat{\mathbf{k}} \\ &= \left[\frac{1}{t+1}\right]_0^1 \hat{\mathbf{i}} + \left[\frac{1}{t^2+1}\right]_0^1 \hat{\mathbf{j}} + \left[\frac{t}{t^2+1}\right]_0^1 \hat{\mathbf{k}} \\ &= \ln(2)\hat{\mathbf{i}} + \frac{\pi}{4}\hat{\mathbf{j}} + \frac{1}{1}(\ln(2))\hat{\mathbf{k}}\end{aligned}$$

3 Partial Derivative

3.1 Function of Several Variables

Definition 3.1.1 (Multivariable Functions). A function of f of n variables is a function that takes any n-tuple $(x_1, \cdots, x_n)$ in the set D to a number in $\mathbb{R}$, where

$$D = \Big\{(x_1, \cdots, x_n) | x_i \in \mathbb{R} \text{ and } f \text{ is defined in } (x_1, \cdots, x_n)\Big\}$$

Example 3.1.1. $f(x, y) = \sqrt{x^2 + y^2 - 4}$: $f : \begin{array}{l} \mathbb{R}^2 \longrightarrow \mathbb{R} \\ (x, y) \longmapsto \text{ a number like } r \end{array}$

Domain of f: all $(x, y) \in \mathbb{R}$ *s.t.* $x^2 + y^2 - 4 \geq 0$. (i.e., Everything exclude the circle centered at the origin with a radius of 2.)

Definition 3.1.2 (Graphs of a Two-Variable Function). The graph of a two-variable function with domain D is the set of all points $(x, y, z) \in \mathbb{R}^3$ *s.t.* $z = f(x, y)$ and $(x, y) \in D$.

Definition 3.1.3 (Vector Functions).

$$\vec{\mathbf{r}} : \begin{array}{l} \mathbb{R} \longrightarrow V_n \\ t \longmapsto \langle f(t),\ g(t),\ h(t), \cdots \rangle \end{array},$$

where V_n is a set of all vectors with n components, and t is a parameter.

Remark. We will only work with V_3, i.e., $\vec{\mathbf{r}} : \begin{array}{l} \mathbb{R} \longrightarrow V_3 \\ t \longmapsto \langle f(t),\ g(t),\ h(t) \rangle \end{array}$.

Theorem 3.1.1. A multivariable function creates a surface in the space. if two surfaces intersect each other, then the intersection identifies a curve.

Example 3.1.2. Find a vector function $\vec{\mathbf{r}}(t)$ that represents the curve of intersection of two surfaces

$$z = \sqrt{x^2 + y^2} \qquad \text{and} \qquad z = 3 + y.$$

Answer.

Solve the system of equation $\begin{cases} x = \sqrt{x^2 + y^2} \\ z = 3 + y \end{cases}$.

Hence,

$$\begin{aligned}\sqrt{x^2+y^2} &= 3+y \\ x^2+y^2 &= (3+y)^2 = y^2+6y+9 \\ x^2 &= 6y+9 \\ y &= \frac{x^2-9}{6} \\ \therefore z =& 3+y = \frac{x^2+0}{6}\end{aligned}$$

Let $x = t$:

$$\vec{\mathbf{r}}(t) = \langle x, t, z\rangle = \left\langle t,\ \frac{t^2-9}{6},\ \frac{t^2+9}{6}\right\rangle$$

□

Example 3.1.3. Do the same for surfaces

$$z = 3x^2+y^2 \qquad \text{and} \qquad y = 5x^2$$

Answer.

Solve the system of equations $\begin{cases} z = 3x^2+y^2 \\ y = 5x^2 \end{cases}$.

$$\therefore 5x^2 = 3x^2+y^2 \implies z = 3x^2+(5x^2)^2 = 3x^2+25x^4$$

Let $x = t$:

$$\vec{\mathbf{r}}(t) = \langle x, t, z\rangle = \left\langle t,\ 5t^2,\ 3t^2+25t^4\right\rangle$$

□

Definition 3.1.4 (Level Curves). The level curve of a two variable function $z = f(x, y)$ is a curve $f(x, y) = k$ (in the xy-plane). That means all values of x and y that have the same value $z = k$.

Theorem 3.1.2 (Application of Level Curve). Given that a point (a, b) is on the level curve of $f(x, y)$ for $k = c$, then we know $f(a, b) = c$.

3.2 Limit and Continuity

Definition 3.2.1 (Limit). For two variable function $z = f(x, y)$, we check limit when $(x, y) \to (a, b)$. Therefore, we can make (x, y) closer to $a(b)$ from infinitely many directions. Therefore,

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if in all directions that (x, y) approaches to (a, b), we have $f(x, y) \to L$.

Definition 3.2.2 (Precise Definition of Limit). $\forall$ given $\varepsilon > 0$, $\exists$ associated $\delta > 0$s.t. if $(x, y) \in D$ and $d((x, y), (a, b)) < \delta \implies d(f(x, y), L) < \varepsilon$, where $d((x, y), (a, b))$ is the distance between (x, y) and (a, b) and is calculated by $\sqrt{(x-a)^2 + (y-b)^2}$.

Example 3.2.1. Consider function $f(x, y) = \dfrac{xy}{x^2 + y^2}$, and identify if it is has a limit at $(0, 0)$ or not.

Answer.

In the direction of x-axis ($y = 0$), we have $f(x, y) = \dfrac{x \cdot 0}{x^2 + 0^2} = 0$ and $\lim\limits_{(x,y)\to(0,0)} f(, y) = 0$ along the x-axis.

In the direction of y-axis ($x = 0$), we have $f(x, y) = 0.$, and $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0$ along the y-axis.

If $y = x$, $f(x, y) = f(x, x) = \dfrac{x^2}{x^2 + x^2} = \dfrac{1}{2}$, and $\lim\limits_{(x,y)\to(0,0)} f(x, y) = \dfrac{1}{2}$ along the line $y = x$.

□

Example 3.2.2. Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^2 + y^2}$.

Answer.

By looking at the graph of the function, we think it has a limit at $(0, 0)$. This is not enough, and later we will be able to say that limit exists by converting it to polar coordinate.

Let $y = mx$:

$$f(x, y) = f(x, mx) = \frac{x^2 \cdot mx}{x^2 + (mx)^2} = \frac{x^3 m}{x^2(1 + m^2)} = \frac{m}{1 + m^2} x$$

$$\therefore \lim_{(x,y)\to(0,0)} f(x, y) = 0 \text{ along the line of } y = mx.$$

□

Example 3.2.3.

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^2} = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^4} = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{3x^3 y}{x^4 + y^4} \text{ D.N.E. } \left(\text{check } \begin{cases} x = 0 \\ y = x \end{cases} \right)$$

Definition 3.2.3 (Continuity). Functions of two-variables is continues at (a, b) if

$$\lim_{(x,y)\to(a,b)} = f(a, b).$$

Example 3.2.4. Find $\lim_{(x,y)\to(1,2)}(x^2y^3 - x^3y^2 + 3x + 2y)$.

Answer.

As $x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial and continuous everywhere, so

$$\lim_{(x,y)\to(1,2)}(x^2y^3 - x^3y^2 + 3x + 2y) = (1)^2(2)^3 - (1)^3(2)^2 + 3(1) + 2(2) = 1.$$

□

Example 3.2.5. $f(x,y) = \dfrac{x^2y}{x^2+y^2}$ is not continuous at $(0,0)$, but

$$g(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \text{ is continuous at } (0,0).$$

3.3 Partial Derivatives

In two-variable functions, we will have partial derivatives f_x (derivative with respect to x) and f_y (derivative with respect to y).

Definition 3.3.1 (Partial Derivative). If $f(x,y)$ is a two variable function, then its partial derivatives are f_x and f_y and is defined as

$$\frac{\partial f}{\partial x} = f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial f}{\partial y} = f_y(x,y) = \lim_{h\to 0}\frac{f(x,y+h) - f(x,y)}{h}$$

Example 3.3.1. Let $f(x,y) = x^3 + x^2y^3 - 2y$ and find $f_x(2,1)$ and $f_y(2,1)$

Answer.

Find $f_x(x,y)$: keep y constant.

$f_x(x,y) = 3x^2 + 2xy^3$

$$\therefore f_x(2,1) = 3(2)^2 + 2(2)(1)^3 = 16$$

Find $f_y(x,y)$: keep x constant.

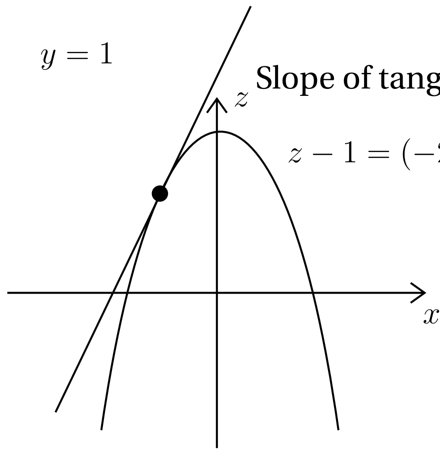
$f_y(x,y) = 3x^2y^2 - 2$

$$\therefore f_y(2,1) = 3(2)^2(1)^2 - 2 = 10$$

□

Example 3.3.2. Let $f(x,y) = 4 - x^2 - 2y^2$. Find $f_x(1,1)$ and interpret the values.

Answer.



$f(1,1) = 4 - 1 - 2 = 1 \implies A(1,1,1)$ lies on $f(x,y)$.

$\dfrac{\partial f}{\partial x} = -2x \implies \dfrac{\partial f}{\partial x}(1,1) = -2$

Let's consider $y = 1$:

The plane $y = 1$ will intersect with $f(x,y)$ at a line $\vec{\mathbf{r}}(t)$.

Solve $\vec{\mathbf{r}}(t)$: $\begin{cases} z = 4 - x^2 - 2y^2 \\ y = 1 \end{cases}$

$$\Rightarrow z = 4 - x^2 - 2 = 2 - x^2$$

$$\therefore \vec{\mathbf{r}}(t) = \langle t, 1, 2 - t^2 \rangle,\ \vec{\mathbf{r}}'(t) = \langle 1, 0, -2t \rangle$$

At point $A(1,1,1)$, $t = 1$.

$\therefore \vec{\mathbf{r}}'(1) = \langle 1, 0, -2 \rangle$, which is a directional vector of the tangent line.

$\therefore$ Tangent line:

$$L:\ x = 1 + t,\ y = 1,\ z = 1 - 2t$$

□

Definition 3.3.2 (Higher Order Partial Derivative).

$$\frac{\partial^2 f}{\partial x \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial y \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Theorem 3.3.1 (Clairaut's Theorem). If f is continuous on a disk D, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

Definition 3.3.3 (Functions With More Than Two Variables). If $U = f(x_1, \cdots, x_n)$, its partial derivative with respect to x_i is

$$\begin{aligned} \frac{\partial f}{\partial y_i} &= \lim_{h \to 0} \frac{f(x_a, \cdots, x_i + h, \cdots, x_n) - f(x_1, \cdots, x_n)}{h} \\ &= \frac{\partial U}{\partial x_i} \end{aligned}$$

3.4 Tangent Plane and Linear Approximation

Theorem 3.4.1 (Tangent Plane). If f has continuous partial derivatives, an equation of the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) is

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

Example 3.4.1. Find the tangent plane of $f(x, y) = 2x^2 + y^2$ at $(1, 1, 3)$.

Answer.

$$\frac{\partial f}{\partial x} = 4x \qquad \frac{\partial f}{\partial y} = 2y$$

$$\therefore \frac{\partial f}{\partial x}(1, 1) = 4 \qquad \frac{\partial f}{\partial y}(1, 1) = 2$$

$\therefore$ Tangent plane at $(1, 1, 3)$:

$$\Pi : z - 3 = 4(x - 1) + 2(y - 3).$$

□

Definition 3.4.1 (Linearization and Linear Approximation). Similar to single variable calculus, we can approximate the value of a function at a point using the tangent line:

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is the **linearization** of $f(x, y)$ at point (a, b):

$$f(x, y) \approx L(x, y)$$

is called the **linear approximation** or the **tangent plane approximation** of f at (a, b).

Definition 3.4.2 (Differentiable Functions). A **differentiable function** is a function that the linear approximation is a good approximation when (x, y) are very close to (a, b).

Theorem 3.4.2 (A sufficient condition for differentiability). If partial derivative $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exists near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Example 3.4.2. Show that function $f(x, y) = \dfrac{\sqrt{x}}{y}$ is differentiable at $(16, 5)$ and use it to approximate $\dfrac{\sqrt{16.02}}{4.96}$.

Answer.

$$f(16,5) = \frac{\sqrt{16}}{5} = \frac{4}{5}; \quad \frac{\partial f}{\partial x} = \frac{1}{2y\sqrt{x}}; \quad \frac{\partial f}{\partial y} = -\frac{\sqrt{x}}{y^2}.$$

$$\therefore \left.\frac{\partial f}{\partial x}\right|_{(16,5)} = \frac{1}{2(5)\sqrt{16}} = \frac{1}{40}; \quad \left.\frac{\partial f}{\partial y}\right|_{(16,5)} = -\frac{\sqrt{16}}{25} = -\frac{4}{25}.$$

As $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exists and is continuous at $(x, y) = (16, 5)$, $f(x, y)$ is differentiable at $(16, 5)$.

Then, the approximation is

$$f(x,y) \approx f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

At $a = 16$ and $b = 5$:

$$\begin{aligned}
\frac{\sqrt{x}}{y} &\approx \frac{4}{5} + \frac{1}{40}(x-16) + \left(-\frac{4}{25}\right)(y-5) \\
&= \frac{4}{5} + \frac{1}{40}x - \frac{2}{5} - \frac{4}{25}y + \frac{4}{5} \\
&= \frac{1}{40} - \frac{4}{25}y + \frac{6}{5}.
\end{aligned}$$

Therefore, $\dfrac{\sqrt{16.02}}{4.96} \approx \dfrac{1}{40}(16.02) - \dfrac{4}{25}(4.96) + \dfrac{6}{5} \approx 0.807$.

□

Definition 3.4.3 (Differentials).

$$\Delta z = \Delta f = f(a + \Delta x, b + \Delta y) - f(a, b)$$

$$\mathrm{d}z = f_x(a,b)\mathrm{d}x + f_y(a,b)\mathrm{d}y$$

Extension (Differentials in Higher Dimensions). Let $U = f(x_1, x_2, \cdots, x_n)$, we have

$$\mathrm{d}U = f_{x_1}(a_1, \cdots, a_n)\mathrm{d}x_1 + f_{x_2}(a_1, \cdots, a_n)\mathrm{d}x_2 + \cdots + f_{x_n}(a_1, \cdots, a_n)\mathrm{d}x_n$$

$$\Delta U = \Delta f = f(x_1 + \Delta x_1, x_2 + \Delta x_2, \cdots, x_n + \Delta x_n) - f(x_a, \cdots, x_n)$$

3.5 The Chain Rule

Theorem 3.5.1 (The Multivariable Chain Rule). Let U be a differentiabel function of n variables $x_1, \cdots, x_n$, and each x_i for $i = 1, \cdots, n$ is a differentiable function of $t_1, \cdots, t_m$. Then, we have

$$\frac{\partial U}{\partial t_i} = \frac{\partial U}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial U}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial U}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

Example 3.5.1. Let $U = x^4y + y^2z^3$ and $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin(t)$. Find thee value of $\frac{\partial U}{\partial s}$ when $r = 2,\ s = 1,\ t = 0$.

Answer.

From the multivariable china rule, we know

$$\frac{\partial U}{\partial s} = \frac{\partial U}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial U}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial U}{\partial x} = 4x^3y; \quad \frac{\partial U}{\partial x} = x^4 + 2yz^3; \quad \frac{\partial U}{\partial x} = 3y^2z^2;$$

$$\frac{\partial x}{\partial s} = re^t; \quad \frac{\partial y}{\partial s} = 2rse^{-t}; \quad \frac{\partial x}{\partial s} = r^2\sin t.$$

$$\therefore \frac{\partial U}{\partial s} = (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2\sin t)$$

When $r = 2,\ s = 1,\ t = 0$, we have

$$x = 2,\ y = 2,\ z = 0.$$

$$\therefore \left.\frac{\partial U}{\partial s}\right|_{(r,s,t)=(2,1,0)} = (4(2)^3(2))(2) + (2^4)(2 \cdot 2) + 0 = 128 + 64 = 192.$$

□

Example 3.5.2. If $z = f(x, y)$ has continuous second order partial derivatives and $x = r^2+s^2$ and $y = 2rs$. Find $\frac{\partial z}{\partial r}$ and $\frac{\partial^2 z}{\partial r^2}$.

Answer.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

Since

$$\frac{\partial}{\partial r} = 2r; \qquad \frac{\partial y}{\partial r} = 2s$$

$$\therefore \frac{\partial z}{\partial r} = 2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}.$$

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) &= \frac{\partial}{\partial r}\left(2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}\right)\\ &= 2\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial x}\right) + 2\frac{\partial}{\partial r}\left(s\frac{\partial z}{\partial y}\right)\\ &= 2\left[\frac{\partial}{\partial r}(r)\cdot\frac{\partial z}{\partial x} + r\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)\right] + 2\left[\frac{\partial}{\partial r}(s)\cdot\frac{\partial z}{\partial y} + s\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)\right]\end{aligned}$$

Notice that $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are functions dependent on x and y, so to find their partial derivatives with respect to r, we need to apply multivariable chain rule again:

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y\partial x}\frac{\partial y}{\partial r}$$
$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}$$

$$\therefore \frac{\partial^2 z}{\partial r^2} = 2\frac{\partial z}{\partial x} + 2r\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y\partial x}\frac{\partial y}{\partial r}\right) + 2s\left(\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\right)$$

□

Theorem 3.5.2 (Implicit Differentiation). If we have two-variable function like $F(x,y) = 0$, where y depends on x, we use the multivariable chain rule to differential the both sides of $F(x,y)$:

$$\frac{\partial F}{\partial x}\cdot\underbrace{\frac{\mathrm{d}x}{\mathrm{d}x}}_{1} + \frac{\partial F}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\frac{\partial F}{\partial x} = -\frac{\partial F}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{F_x}{F_y}$$

Example 3.5.3. Find y' if $x^3 + y^3 = 6xy$

Answer.

Method1 Applying the formula:

$$F_x = 3x^2 - 6y$$
$$F_y = 3y^2 - 6x$$
$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x}$$

Medthod2 Find derivatives of the both sides:

$$
\begin{aligned}
x^3 + y^3 - 6xy &= 0 \\
3x^2 + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} - 6y - 6x\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(3y^2 - 6x)\frac{\mathrm{d}y}{\mathrm{d}x} &= 6y - 3x^2 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6y - 3x^2}{3y^2 - 6x}
\end{aligned}
$$

□

Theorem 3.5.3 (Multivariable Implicit Differentiation). If $z = f(x, y)$, consider a function

$$F(x, y, z) = F(x, y, f(x, y))$$

Then, by the multivariable chain rule, we differentiate both sides of $F(x, y, f(x, y)) = 0$:

$$\frac{\partial F}{\partial x}\underbrace{\frac{\mathrm{d}x}{\mathrm{d}x}}_{1} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \implies \frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}$$

Similarly, we have

$$\frac{\partial F}{\partial y}\underbrace{\frac{\mathrm{d}y}{\mathrm{d}y}}_{1} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0 \implies \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}$$

Example 3.5.4. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

Answer.

In order to find $\frac{\partial z}{\partial x}$, differentiate both sides with respect to x:

$$
\begin{aligned}
3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} &= 0 \\
(3z^2 + 6xy)\frac{\partial z}{\partial x} &= -(3x^2 + 6yz) \\
\frac{\partial z}{\partial x} &= -\frac{3x^2 + 6yz}{3z^2 + 6xy}\left(= -\frac{x^2 + 2yz}{z^2 + 2xy}\right)
\end{aligned}
$$

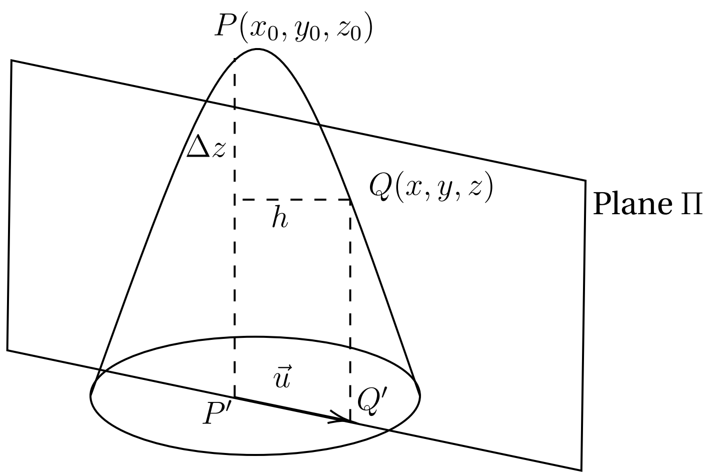
In order to find $\frac{\partial z}{\partial y}$, differentiate both sides with respect to y:

$$\begin{aligned} 3y^2 + 3z^2\frac{\partial z}{\partial y} + 6xz + 6xy\frac{\partial z}{\partial y} &= 0 \\ (3z^2 + 6xy)\frac{\partial z}{\partial y} &= -(3y^2 + 6xz) \\ \frac{\partial z}{\partial y} &= -\frac{3y^2 + 6xz}{3z^2 + 6xy}\left(= -\frac{y^2 + 2xz}{z^2 + 2xy}\right) \end{aligned}$$

□

3.6 Directional Derivatives and Gradient

To formally study directional derivatives, we start from the ideas of it. We want to study the change of $z = f(x, y)$ in the direction of the unit vector $\vec{\mathbf{u}} = \langle a, b\rangle = a\hat{\mathbf{i}} + \hat{\mathbf{j}}$. ($\sqrt{a^2 + b^2} = 1$). We intersect surface $z = f(x, y)$ with plane Π that passes through the point $P(x_0, y_0, z_0)$ vertically and in the direction of vector $\vec{\mathbf{u}} = \langle a, b\rangle$.



So, we have

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + h, y_0 + h) - f(x_0, y_0)}{h}$$

Definition 3.6.1 (Directional Derivative). The directional derivative of f at (x_0, y_0) in the direction of a vector $\vec{\mathbf{u}} = \langle a, b\rangle$ is defined as

$$D_{\vec{\mathbf{u}}}f(x_0 y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if the limit exists.

Now, let $g(h) = f(x_0 + ha, y_0 + hb)$, then we have

$$D_{\vec{\mathbf{u}}} f(x_0, y_0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = g'(0)$$

To find $g'(h)$, we use the multivariable chain rule:

$$g'(h) = \frac{\partial g}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}h} + \frac{\partial g}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}h} \quad \text{where } \begin{cases} x = x_0 + ha \\ y = y_0 + hb \end{cases}.$$

From $\begin{cases} x = x_0 + ha \\ y = y_0 + hb \end{cases}$, we have $\dfrac{\partial x}{\partial h} = a$ and $\dfrac{\partial y}{\partial h} = b$.

$$\begin{aligned} \therefore g'(h) &= \frac{\partial g}{\partial x} \cdot a + \frac{\partial g}{\partial y} \cdot b \\ &= a \cdot \frac{\partial f}{\partial x} + b \cdot \frac{\partial f}{\partial y} \qquad \Big[g(h) \text{ is in fact } f(x, y) \Big] \end{aligned}$$

When $h \to 0$,

$$g'(0) = a \cdot f_x(x_0, y_0) + b \cdot f_y(x_0, y_0)$$

$$\begin{aligned} \therefore D_{\vec{\mathbf{u}}} f(x_0, y_0) &= a \cdot f_x(a_0, y_0) + b \cdot f_y(x_0, y_0) \\ &= \langle a, b \rangle \cdot \langle f_x(a_0, y_0), f_y(x_0, y_0) \rangle \end{aligned}$$

Theorem 3.6.1 (Directional Derivative in Dot Product).

$$D_{\vec{\mathbf{u}}} f(x_0, y_0) = \vec{\mathbf{u}} \cdot \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle = \vec{\mathbf{u}} \cdot \nabla f(x_0, y_0)$$

Definition 3.6.2 (Gradient Vector). A gradient vector of f is a vector function defined as

$$\nabla f(x, y) = \frac{\partial f}{\partial x} \hat{\mathbf{i}} + \frac{\partial f}{\partial y} \hat{\mathbf{j}}.$$

The notation "∇" is called nabla.

Extension. If f is a function as $f(x_1, \cdots, x_n)$, then

$$\nabla f = \langle f_{x_1}, f_{x_2}, f_{x_3} \cdots, f_{x_n} \rangle.$$

Theorem 3.6.2 (Properties of Gradient). From the dot product definition of directional vector, we know that

$$D_{\vec{\mathbf{u}}} f = \nabla f \cdot \vec{\mathbf{u}}.$$

Then, if θ is the angle between ∇f and $\vec{\mathbf{u}}$, we have

$$D_{\vec{\mathbf{u}}} f = |\nabla f||\vec{\mathbf{u}}|\cos\theta.$$

Thus,

$$\max D_{\vec{\mathbf{u}}} f = |\nabla f||\vec{\mathbf{u}}| \text{ when } \theta = 0$$

(or, the vector $\vec{\mathbf{u}}$ is in the direction of ∇f.) Since $\vec{\mathbf{u}}$ is a unit vector, $|\vec{\mathbf{u}}| = 1$. So when $\vec{\mathbf{u}}$ is in the same direction of ∇f, we have

$$\max D_{\vec{\mathbf{u}}} f = |\nabla f|.$$

On the other hand, if $\vec{\mathbf{u}}$ and ∇f are in the opposite direction, we have $\theta = \pi$ and $\cos\theta = \cos(\pi) = -1$.

$$\therefore \min D_{\vec{\mathbf{u}}} f = |\nabla f||\vec{\mathbf{u}}|\cos\theta = -|\nabla f|$$

Extension. If $\vec{\mathbf{u}}$ is a unit vector and $\vec{\mathbf{u}} = \langle a, b\rangle$ and f has continuous second partial derivatives, then

$$D^2_{\vec{\mathbf{u}}} f = f_{xx}a + 2f_{xy}ab + f_{yy}b.$$

Example 3.6.1. If $f(x, y) = xe^y$, then

1. Find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q\left(\frac{1}{2}, 2\right)$.

   Answer.

$$\frac{\partial f}{\partial x} = e^y; \quad \frac{\partial f}{\partial y} = xe^y; \quad \overrightarrow{PQ} = \left\langle \frac{1}{2} - 2, 2 - 0\right\rangle = \left\langle -\frac{3}{2}, 2\right\rangle; \quad \left|\overrightarrow{PQ}\right| = \sqrt{\frac{9}{4} + 4} = \frac{5}{2}$$

$$\therefore \vec{\mathbf{u}} = \left\langle -\frac{3}{2}\cdot\frac{2}{5}, 2\cdot\frac{2}{5}\right\rangle = \left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle; \quad \nabla f = \langle e^y, xe^y\rangle.$$

   Therefore,

$$D_{\vec{\mathbf{u}}} f = \nabla f \cdot \vec{\mathbf{u}} = \langle e^y, xe^y\rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle = -\frac{3}{5}e^y + \frac{4}{5}xe^y.$$

   At point $P(2, 0)$,

$$D_{\vec{\mathbf{u}}} f(2, 0) = -\frac{3}{5}e^0 + \frac{4}{5}\cdot 2\cdot e^0 = -\frac{3}{5} + \frac{8}{5} = 1.$$

   □

2. In what direction does f have the maximum rate of change? What is this maximum rate of change?

   Answer.

$$\nabla f(2,0) = \langle e^0, 2e^0 \rangle = \langle 1, 2 \rangle$$

Hence, in direction $\nabla f = \langle 1, 2 \rangle$, f has the maximum rate of change. The maximum rate of change is $|\nabla f(2,0)| = \sqrt{5}$.

□

Theorem 3.6.3 (Gradient and Tangent Plane). The equation of the tangent plane for the surface $z = f(x,y)$ at the point $P(x_0, y_0, z_0) =$ is given by:

$$\nabla f \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

or (for implicit functions)

$$\frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) + \frac{\partial f}{\partial z}(z - z_0) = 0.$$

The normal line of the plane is given by

$$\frac{x - x_0}{f_x(x_0, y_0, z_0)} = \frac{y - y_0}{f_y(x_0, y_0, z_0)} = \frac{z - z_0}{f_z(x_0, y_0, z_0)}.$$

Remark (Gradient and Multivariable Chain Rule). If $F(x,y,z) = k$ and x, y, z are dependent of t, then we differentiate both sides with respect to t to get:

$$\begin{aligned}
\frac{\partial F}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial F}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial F}{\partial z} \cdot \frac{\mathrm{d}z}{\mathrm{d}t} &= 0 \\
\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle \left\langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t} \right\rangle &= 0 \\
\nabla F \cdot \left\langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t} \right\rangle &= 0
\end{aligned}$$

Theorem 3.6.4 (Graphical Interpretation of Gradient Vector). In general, the gradient vector at P, $\nabla F(x_0, y_0, z_0)$ is perpendicular to the tangent vector $\vec{\mathbf{r}}'(t_0)$ to any curve C that passes through the point P on the surface S. Similar properties hold on level curves.

3.7 Maximum and Minimum Values

Definition 3.7.1 (Local Maximum and Local Minimum). A function $f(x,y)$ has a **local maximum** at point (a,b) if $\forall (x,y)$ near point (a,b), we have $f(x,y) \leq f(a,b)$. The function $f(x,y)$ has a **local minimum** at point (a,b) if $\forall (x,y)$ near point (x,y), we have $f(x,y) \geq f(a,b)$.

Remark. "near point (a,b)" refers to a disk centered at (a,b).

Definition 3.7.2 (Absolute Maximum and Absolute Minimum). If the equalities $f(x, y) \leq f(a, b)$ and $f(x, y) \geq f(a, b)$ holds for any (x, y) in the domain of $f(x, y)$, then we call them **absolute maximum** or **absolute minimum**.

Theorem 3.7.1. If f has local maximum or minimum at (a, b), and the first order partial derivatives of f exist at (a, b), then $f_x(a, b)$ and $f_y(a, b)$ are equal to 0. In other words,

$$\nabla f(a, b) = 0.$$

Corollary 3.1. As a result of Theorem 3.7.1, the equation of the tangent plane at (a, b) is

$$\begin{aligned} z - \overbrace{f(a,b)}^{z_0} &= \overbrace{f_x(a,b)}^{0}(x-a) + \overbrace{f_y(a,b)}^{0}(y-b) \\ z - z_0 &= 0. \end{aligned}$$

In other words, the tangent plane is horizontal.

Definition 3.7.3 (Critical Points). A point (a, b) is called the **critical point** if $f_x(a, b) = 0$ and $f_y(a, b) = 0$ or if one of the partial derivatives does not exist.

Remark. At a critical point, we may have maximum or minimum or neither (saddle point).

Definition 3.7.4 (Determinant). The determinant (Δ or D) is defined as

$$\begin{aligned} D &= \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \\ &= f_{xx}f_{yy} - f_{xy}f_{yx} \\ &= f_{xx}f_{yy} - (f_{xy})^2. \end{aligned}$$

Theorem 3.7.2 (Second Derivative Test). Let (a, b) be a critical point and second partial derivatives of f (i.e., f_{xx}, f_{xy}, f_{yx}, f_{yy}) are continuous on a disk centered at (a, b). Then

1. If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
2. If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
3. If $D < 0$, then $f(a, b)$ is not a local maximum or local minimum, and it is called a **saddle point**.

Remark. At saddle points, the tangent plane will intersect with the surface of f.

Example 3.7.1. For function $f(x, y) = 4 + x^3 + y^3 - 3xy$. Check it $f(x, y)$ has local maximum, local minimum, and saddle points.

Answer.

$$\frac{\partial f}{\partial x} = 3x^3 - 3y; \qquad \frac{\partial f}{\partial y} = 3y^2 - 3x$$

Solve $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 3x^2 - 3y = 0 & ① \\ 3y^2 - 3x = 0 & ② \end{cases}$.

From ①: $y = x^2$.

Substitute $y = x^2$ to ②:

$$\begin{aligned} 3\left(x^2\right)^2 - 3x &= 0 \\ x^4 - x &= 0 \\ x(x^3 - 1) &= 0 \Longrightarrow x = 0 \text{ or } x = 1 \end{aligned}$$

$$\therefore y = 0^2 = 0 \qquad \text{or} \qquad y = 1^2 = 1$$

$$\therefore \begin{cases} x = 0 \\ y = 0 \end{cases} \text{or} \begin{cases} x = 1 \\ y = 1 \end{cases}$$

i.e., Critical points are at $(0, 0)$ and $(1, 1)$.

Find D:

$$D = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix} = 36xy - 9.$$

Apply the second derivative test:

1. $D(0, 0) = -9 < 0 \Longrightarrow (0, 0)$ is a saddle point.
2. $D(1, 1) = 36 - 9 = 27 > 0$ and $\frac{\partial^2 f}{\partial x^2} = 6(1) > 0 \Longrightarrow (1, 1)$ is a local minimum.

□

Theorem 3.7.3 (Extreme Value Theorem, EVT). We are expanding the Extreme Value Theorem from a single variable version to a multivariable version:

1. Single Variable Version: any continuous function on a closed interval I has a maximum or minimum value in that interval I.
2. Multivariable Version: For a multivariable function $f(x_1, \cdots, x_n)$ on a **closed and bounded** region D in $\mathbb{R}^n$. f has both maximum and minimum values in that region.

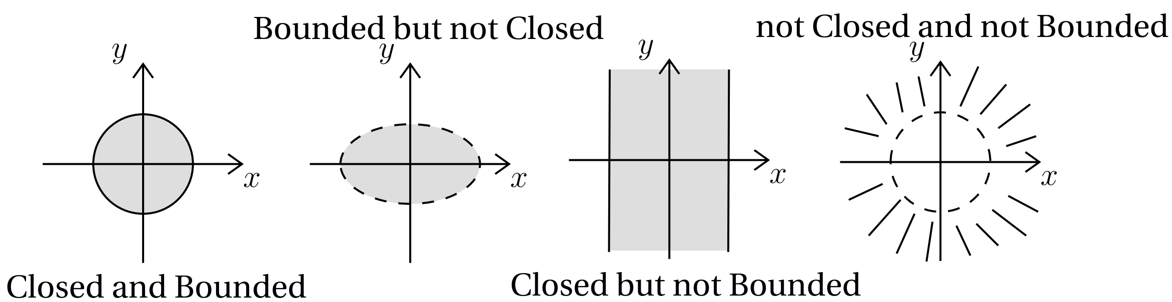
Definition 3.7.5 (Bounded Region). D is bounded if there exists some ball

$$x_1^2 + x_2^2 + \cdots + x_n^2 \leq R^2$$

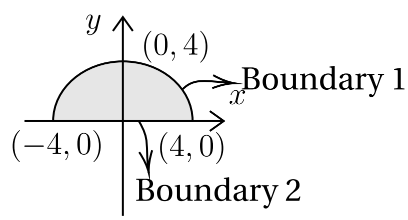
that contains D.

Definition 3.7.6 (Closed Region). Closed region D is a region that includes the boundaries.

Example 3.7.2 (Bounded and Closed Region). The following are examples of closed and bounded regions.



Example 3.7.3. Find the extreme values of the function $f(x, y) = x^2+2y^2-x^2y$ on the following region:



Answer.
We can write the region D as the following set:

$$D = \{(x, y) \mid x^2 + y^2 \leq 6,\ y \geq 0\}.$$

$\boxed{\text{Step } 1}$ Find the critical points of the function that are inside the boundary (interior to the boundary).

$$f(x, y) = x^2 + 2y^2 - x^2y \Rightarrow \nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 2x - 2xy, 4y - x^2 \rangle.$$

Set $\nabla f(x, y) = 0$: $\begin{cases} 2x - 2xy &= 0 \quad ① \\ 4y - x^2 &= 0 \quad ②. \end{cases}$

From ②: $y = \dfrac{x^2}{4}$. Substitute this result into ①:

$$2x - 2x \cdot \frac{x^2}{4} = 0$$
$$2x - \frac{1}{2}x^3 = 0 \Rightarrow x\left(2 - \frac{1}{2}x^2\right) = 0$$

$$\therefore \begin{cases} x &= 0 \\ y &= 0 \end{cases} \text{ or } \begin{cases} x^2 &= 4 \\ y &= 1 \end{cases}$$

$$\Rightarrow \begin{cases} x &= 0 \\ y &= 0 \end{cases} \text{ or } \begin{cases} x &= 2 \\ y &= 1 \end{cases} \text{ or } \begin{cases} x &= -2 \\ y &= 1 \end{cases}$$

All the points $(0,0)$, $(2,1)$, and $(-2,1)$ are inside the boundary.

Step 2 Check the boundaries for maximum and minimum.

Check Boundary 1: $x^2 + y^2 = 16, \quad 0 \le y \le 4.$

$$\begin{aligned} f(x,y) = x^2 + 2y^2 - x^2y &= 16 + y^2 - (16 - y^2)y \\ &= 16 + y^2 - 16y + y^3 \\ f(y) &= y^3 + y^2 - 16y + 16 \rightarrow \text{ one variable function} \end{aligned}$$

$$f'(y) = 3y^2 + 2y - 16 = 0 \qquad y = -\frac{8}{3}, \quad y = 2.$$

Since $0 \le y \le 4$, $y = 2$.

When $y = 2$, $x = \pm\sqrt{16 - 4} = \pm 2\sqrt{3}$.

$$f(y) = 2^3 + 2^2 - 16(2) + 16 = 8 + 4 - 32 + 16 = -4$$

When $y = 4$, $x = 0$.

$$f(x,y) = 16 + 16 - 64 + 64 = 32$$

When $y = 0$, $x = \pm 4$.

$$f(x,y) = 16 \rightarrow \text{ (not a extreme value)}$$

Hence, we have $-4 \le f(x,y) \le 32$ on Boundary 1.

Check boundary 2: $-4 \le x \le 4, \quad y = 0.$

$$f(x,y) = x^2 + 2y^2 - x^2y = x^2$$

Since $0 \le x^2 \le 16, \quad 0 \le f(x,y) \le 16.$

Step 3 List all the points and values:

Point	Value
$(0,0)$	$f(0,0) = 0$
$(2,1)$	$f(2,1) = 2$
$(-2,1)$	$f(-2,1) = 2$
$(2\sqrt{3},2)$	$f(2\sqrt{3},2) = -4$
$(-2\sqrt{3},2)$	$f(-2\sqrt{3},2) = -4$
$(0,4)$	$f(0,4) = 32$

Hence, minimum occurs at $(2\sqrt{3},2)$ and $(-2\sqrt{3},2)$, and the function value is -4 at minimum. The maximum occurs at $(0,4)$, and the function value is 32 at maximum.

□

3.8 Lagrange Multiplier

Definition 3.8.1 (Optimization). Find minimum or maximum values of a function subject to constrains.

Remark. The constrains can be an equality or an inequality.

Definition 3.8.2 (Objective Function). The function f we are working with is called the **objective function** or **cost function**.

Definition 3.8.3 (Linear and Non-Linear Optimization). If the objective function is linear, the process is called **linear programming** or **linear optimization**. If the objective function is not linear, the process if called **non-linear optimization**.

Theorem 3.8.1 (Lagrange Multiplier). The minimum or maximum value of $f(x_1, \cdots, x_n)$ subject to the condition $g(x_1, \cdots, x_n) = k$, where f and g are differentiable, occur when the gradient vectors, ∇f and ∇g, are parallel. That is,

$$\nabla f(x_1, \cdots, x_n) = \lambda \nabla g(x_1, \cdots, x_n)$$

for some λ.

Extension (Lagrange Multiplier with Multiple Constrains). If we have two constrains $g(x_1, \cdots, x_n) = k$ and $h(x_1, \cdots, x_n) = m$, then the minimum or maximum value of $f(x_1, \cdots, x_n)$ occurs at

$$\nabla f(x_1, \cdots, x_n) = \lambda \nabla g(x_1, \cdots, x_n) + \mu \nabla h(x_1, \cdots, x_n)$$

for some λ and μ.

Example 3.8.1. Maximize $f(x, y) = xy$ on the curve $x^2 + y^2 = 4$.

Answer.

In this example, $f(x, y) = xy$, $g(x, y) = x^2 + y^2$, and $k = 4$. Then,

$$\nabla f(x, y) = \langle y, x \rangle \qquad \nabla g(x, y) = \langle 2x, 2y \rangle.$$

Attempt to solve $\nabla f(x, y) = \lambda \nabla g(x, y)$:

$$\langle y, x \rangle = \lambda \langle 2x, 2y \rangle.$$

So, we have $\begin{cases} y = 2\lambda x & \text{①} \\ x = 2\lambda y & \text{②} \end{cases}$

Substitute ① into ② we have $x = 2\lambda(2\lambda x)$, or $x = 4\lambda^2 x$.

Divide x on both sides of the equation, we have $4\lambda^2 = 1$ or $\lambda^2 = \frac{1}{4}$. Hence, $\lambda = \pm\frac{1}{2}$.

$\boxed{\lambda = \frac{1}{2}}$: $y = 2\left(\frac{1}{2}\right) = x$

Substitute $y = x$ into $x^2 + y^2 = 4$: $2x^2 = 4$, or $x^2 = 2$. So $x = \pm\sqrt{2}$.

Hence, critical points when $\lambda = \frac{1}{2}$: $(\sqrt{2}, \sqrt{2})$ or $(-\sqrt{2}, -\sqrt{2})$.

The values of function are $f(\sqrt{2}, \sqrt{2}) = \sqrt{2} \cdot \sqrt{2} = 2$ and $f(-\sqrt{2}, -\sqrt{2}) = (-\sqrt{2})(-\sqrt{2})$.

$\boxed{\lambda = -\frac{1}{2}}$: $y = 2\left(-\frac{1}{2}\right)x = -x$.

Substitute $y = -x$ int $x^2 + y^2 = 4 : 2x^2 = 4$ and $x = \pm\sqrt{2}$.

Hence, critical points are $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$.

The respective values of the function are $f(\sqrt{2}, -\sqrt{2}) = -2$ and $f(-\sqrt{2}, \sqrt{2}) = -2$.

Hence, the maximum occurs at $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$, with the maximum value of 2. and the minimum occurs at $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$, with the minimum value of -2.

□

Extension (Lagrange Multiplier with an Inequality Constrain). If we are having an inequality constrain, we need to check if any critical points of $\nabla f = 0$ satisfies the inequality, if so, the critical points from $\nabla f = 0$ will be the maximum or minimum point for this optimization. If we do not have any critical points of $\nabla f = 0$, critical points calculated from the Lagrange Multiplier will be the maximum or minimum point for the optimization.

4 Multiple Integrals

4.1 Double Integral Over Rectangles

Definition 4.1.1 (Double Integral). Suppose $f(x, y)$ is a two-variable function, then the double integral of it over rectangles is defined by

$$\iint_R f(x, y)\,\mathrm{d}A = \lim_{m,n\to\infty} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

if the limit exists.

Theorem 4.1.1. If $f(x, y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y)\,\mathrm{d}A$$

Example 4.1.1. Approximate the volume of $f(x, y) = x^2 y$ when $R = [0, 2] \times [0, 1]$. Use midpoint approximation and $m = n = 2$.

Answer.

We can compute the following (x, y) points that are used for the approximation:

$$(x_{11}, y_{11}) = \left(\frac{1}{2}, \frac{1}{4}\right) \quad (x_{12}, y_{12}) = \left(\frac{1}{2}, \frac{3}{4}\right) \quad (x_{21}, y_{21}) = \left(\frac{3}{2}, \frac{1}{4}\right) \quad (x_{22}, y_{22}) = \left(\frac{3}{2}, \frac{3}{4}\right)$$

We can also compute the value of ΔA :

$$\Delta A = \Delta x \cdot \Delta y = 1 \times \frac{1}{2} = \frac{1}{2}.$$

Hence, we can approximate the volume:

$$\begin{aligned}
V &\approx \Delta A\Big[f(x_{11}, y_{11}) + f(x_{12}, y_{12}) + f(x_{21}, y_{21}) + f(x_{22}, y_{22})\Big] \\
&= \frac{1}{2}\left[f\left(\frac{1}{2}, \frac{1}{4}\right) + f\left(\frac{1}{2}, \frac{3}{4}\right) + f\left(\frac{3}{2}, \frac{1}{4}\right) + f\left(\frac{3}{2}, \frac{3}{4}\right)\right] \\
&= \frac{1}{2}\left[\left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)^2 \cdot \left(\frac{3}{4}\right) + \left(\frac{3}{2}\right)^2 \cdot \left(\frac{1}{4}\right) + \left(\frac{3}{2}\right)^2 \cdot \left(\frac{3}{4}\right)\right] \\
&= \frac{1}{2}\left(\frac{1}{4} + \frac{9}{4}\right) \\
&= \frac{10}{8} = \frac{5}{4}.
\end{aligned}$$

□

Theorem 4.1.2 (Calculating Double Integrals). In order to compute the double integral on $R = [a, b] \times [c, d]$:

$$\iint_R f(x, y)\, \mathrm{d}A$$

1. First, we hold x fixed and find the integral

$$A(x) = \int_c^d f(x, y)\, \mathrm{d}y$$

   The result is an expression on x is called the integration with respect to y.

2. Then, we find the integral

$$V = \int_a^b A(x)\, \mathrm{d}x = \int_a^b \left[\int_c^d f(x, y)\, \mathrm{d}y\right] \mathrm{d}x$$
$$\int_a^b \int_c^d f(x, y)\, \mathrm{d}y\mathrm{d}x$$

Theorem 4.1.3 (Fubini's Theorem). Suppose f is a continuous function of x and y on the rectangle $R = \{(a, y) \mid a \leq x \leq b, c \leq y \leq d\}$. Then,

$$\iint_R f(x, y)\, \mathrm{d}A = \int_a^b \int_c^d f(x, y)\, \mathrm{d}y\mathrm{d}x = \int_c^d \int_a^b f(x, y)\, \mathrm{d}x\mathrm{d}y.$$

Example 4.1.2. Evaluate $\displaystyle\int_0^3 \int_1^2 x^2 y\, \mathrm{d}y\mathrm{d}x.$

Answer.

$$\begin{aligned}
\int_0^3 \int_1^2 x^2 y\, \mathrm{d}y\mathrm{d}x &= \int_0^3 \left[\frac{1}{2}x^2 y^2\right]_1^2 \mathrm{d}x \\
&= \int_0^3 \left(\frac{1}{2}(4)x^2 - \frac{1}{2}x^2\right) \mathrm{d}x \\
&= \int_0^3 \frac{3}{2}x^2\, \mathrm{d}x \\
&= \left[\frac{1}{3} \cdot \frac{3}{2}x^3\right]_0^3 = \frac{1}{2}(27) = \frac{27}{2}
\end{aligned}$$

□

Example 4.1.3. Evaluate the double integral

$$\iint_R y \sin(xy)\, \mathrm{d}A, \qquad \text{where } R = [1, 2] \times [0, \pi].$$

Answer.
From the Fubini's Theorem,

$$\iint_R y\sin(xy)\,\mathrm{d}A = \int_1^2\int_0^\pi y\sin(xy)\,\mathrm{d}y\mathrm{d}x = \int_0^\pi\int_1^2 y\sin(xy)\,\mathrm{d}x\mathrm{d}y$$

Let $u = xy$, then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = y$, which is $\mathrm{d}u = y\mathrm{d}x.$

$$\begin{aligned}
\therefore \int_0^\pi\int_1^2 y\sin xy\,\mathrm{d}x\mathrm{d}y &= \int_0^\pi\int_y^{2y}\sin(u)\,\mathrm{d}u\mathrm{d}y \\
&= \int_0^\pi [-\cos(u)]_y^{2y}\,\mathrm{d}y \\
&= -\int_0^\pi \cos(2y) - \sin(y)\,\mathrm{d}y \\
&= -\left[\frac{1}{2}\sin(2y) - \sin(y)\right]_0^\pi \\
&= -\left(\frac{1}{2}\Big(\sin(2\pi) - \sin(0)\Big) - \Big(\sin(\pi) - \sin(0)\Big)\right) \\
&= 0
\end{aligned}$$

□

Theorem 4.1.4. For a double integral $f(x,y) = g(x)\cdot h(x)$ on the rectangle $R = [a,b]\times[c,d]$,

$$\iint_R g(x)\cdot h(x)\,\mathrm{d}A = \int_a^b g(x)\,\mathrm{d}x\cdot\int_c^d h(x)\,\mathrm{d}y$$

Example 4.1.4. Evaluate the double integral

$$\iint_R \sin(x)\cos(y)\,\mathrm{d}A, \qquad \text{where } R = \left[0,\frac{\pi}{2}\right]\times\left[0,\frac{\pi}{2}\right]$$

Answer.
By the Fubini's Theorem,

$$\begin{aligned}
\iint_R \sin(x)\cos(y)\,\mathrm{d}A &= \int_0^{\pi/2}\sin(x)\,\mathrm{d}x\cdot\int_0^{\pi/2}\cos(y)\,\mathrm{d}y \\
&= \Big[-\cos x\Big]_0^{\pi/2}\cdot\Big[\sin(y)\Big]_0^{\pi/2} \\
&= \left[-\cos\left(\frac{\pi}{2}\right) + \cos(0))\right]\cdot\left[\sin\left(\frac{\pi}{2}\right) - \sin(0)\right] \\
&= (1)(1) = 1
\end{aligned}$$

□

Definition 4.1.2 (Average Value). In two-variable functions, then the average value of f on the rectangle $R = [a,b] \times [c,d]$, f_{ave} is given by

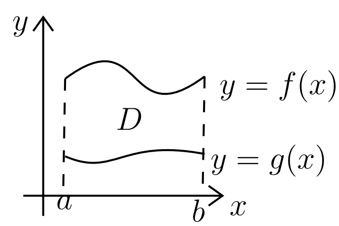
$$f_{\text{ave}} = \frac{\displaystyle\iint_R f(x,y)\,\mathrm{d}A}{A(R)} \qquad \text{or} \qquad \iint_R f(x,y)\,\mathrm{d}A = A(R) \cdot f_{\text{ave}}.$$

4.2 Double Integral Over General Region

Definition 4.2.1 (Double Integral Over a General Region). Furthering the definition of double integral over a rectangle, we use the notation $\iint_D f(x,y)\,\mathrm{d}A$ to represent a double integral of $f(x,y)$ over a general region D.

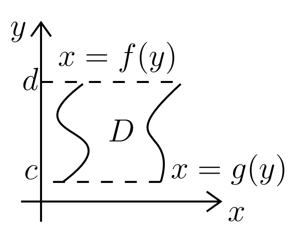
Theorem 4.2.1 (Two Fundamental Types of Region *D*). Here, we discuss two fundamental types of region D, which includes one variable to be dependent on the other.

1. $D = \{(x,y) \mid a < x < b,\ g(x) \le y \le f(x)\}$



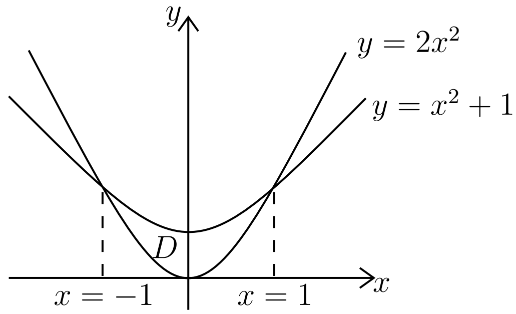
$$\iint_D f(x,y)\,\mathrm{d}A = \int_{g(x)}^{f(x)} \int_a^b f(x,y)\,\mathrm{d}x\mathrm{d}y$$

2. $D = \{(x,y) \mid f(y) \le x \le g(y),\ c < y < d\}$



$$\iint_D f(x,y)\,\mathrm{d}A = \int_{f(y)}^{g(y)} \int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x$$

Example 4.2.1. Find $\iint_D x + 2y\,\mathrm{d}A$, where D is the region bounded by $y = 2x^2$ and $y = x^2 + 1$.



Answer.

$$\iint_D f(x,y)\,\mathrm{d}A = \int_{-1}^{1}\int_{2x^2}^{x^2+1} x + 2y\,\mathrm{d}y\mathrm{d}x = \int_{-1}^{1}\left[xy + y^2\right]_{2x^2}^{x^2+1}\mathrm{d}x$$
$$= \int_{-1}^{1} x\left(x^2+1\right) + \left(x^2+1\right)^2 - x\left(2x^2\right) - \left(2x^2\right)^2\,\mathrm{d}x$$

$$\therefore \iint_D x + 2y\,\mathrm{d}A = \int_{-1}^{1} x\left(x^2+1\right) + \left(x^2+1\right)^2 - x\left(2x^2\right) - \left(2x^2\right)^2\,\mathrm{d}x$$
$$= \int_{-1}^{1} -3x^4 - x^3 + 2x^2 + x + 1\,\mathrm{d}x$$
$$= \left[-\frac{3}{5}x^5 - \frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + x\right]_{-1}^{1}$$
$$= -\frac{3}{5} - \frac{1}{4} + \frac{2}{3} + \frac{1}{2} + 1 - \left(\frac{3}{5} - \frac{1}{4} - \frac{2}{3} + \frac{1}{2} - 1\right)$$
$$= -\frac{6}{5} + \frac{4}{3} + 2$$
$$= \frac{32}{15}$$

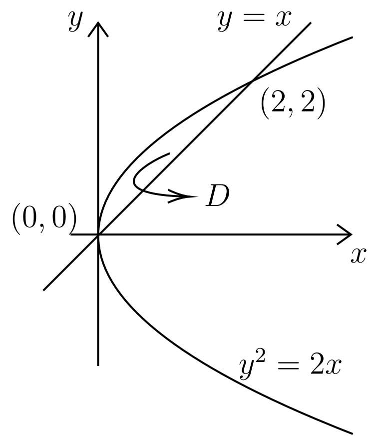
□

Theorem 4.2.2.

$$\iint_D 1\,\mathrm{d}A = A(D) = \text{Area of } D.$$

Example 4.2.2. Sketch the region D in the xy-plane bounded by $y^2 = 2x$ and $y = x$. Find the area of D.

Answer.



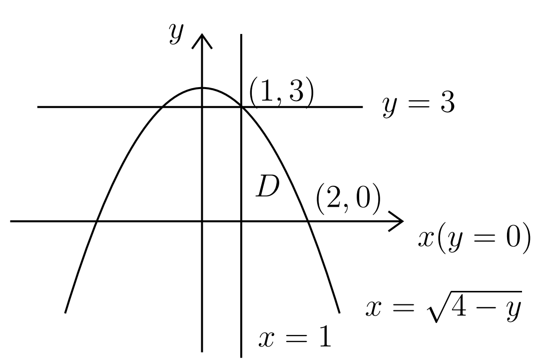
$$\begin{aligned}\text{Area of } D &= \iint_D 1 \, \mathrm{d}A = \iint_D 1 \, \mathrm{d}y\mathrm{d}x \\ &= \int_0^2 \int_x^{\sqrt{2x}} 1 \, \mathrm{d}y\mathrm{d}x \\ &= \int_0^2 (\sqrt{2x} - x) \, \mathrm{d}x \\ &= \left[\sqrt{2}\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2\right]_0^2 \\ &= \left(\frac{2\sqrt{2}}{3}\left(\sqrt{2}\right)^3 - \frac{1}{2}(4) - 0\right) \\ &= \frac{8}{3} - 2 = \frac{2}{3}\end{aligned}$$

□

Example 4.2.3. Given $\int_0^3 \int_1^{\sqrt{4-y}} x + y \, \mathrm{d}x\mathrm{d}y.$

(a) Sketch the region.

Answer.



□

(b) Interchange the order.

Answer.

$$\int_0^3 \int_1^{\sqrt{4-y}} x + y \, \mathrm{d}x\mathrm{d}y = \int_1^2 \int_0^{4-x^2} x + d \, \mathrm{d}y\mathrm{d}x$$

□

(c) Evaluate the integral.

Answer.

$$\begin{aligned}
\int_1^2 \int_0^{4-x^2} x + y \, \mathrm{d}y\mathrm{d}x &= \int_1^2 \left[xy + \frac{1}{2}y^2 \right]_0^{4-x^2} \mathrm{d}x \\
&= \int_1^2 \left[x(4-x^2) + \frac{1}{2}\left(4-x^2\right)^2 \right] \mathrm{d}x \\
&= \int_1^2 \left(4x - x^3 + \frac{1}{2}\left(16 + x^4 - 8x^2\right) \right) \mathrm{d}x \\
&= \int_1^2 \frac{1}{2}x^4 - x^3 - 4x^2 + 4x + 8 \, \mathrm{d}x \\
&= \left[\frac{1}{2} \cdot \frac{1}{5}x^5 - \frac{1}{4}x^4 - 4 \cdot \frac{1}{3}x^3 + 4 \cdot \frac{1}{2}x^2 + 8x \right]_1^2 \\
&= \frac{1}{10}\left(2^5 - 1\right) - \frac{1}{4}\left(2^4 - 1\right) - \frac{4}{3}\left(2^3 - 1\right) + 2\left(2^2 - 1\right) + 8(2-1) \\
&= \frac{31}{10} - \frac{15}{4} - \frac{28}{3} + 6 + 8 \\
&= \frac{241}{60}
\end{aligned}$$

□

Theorem 4.2.3. Properties of Double Integral:

1. $$\iint_D \left[f(x,y) + g(x,y) \right] \mathrm{d}A = \iint_D f(x,y) \, \mathrm{d}A + \iint_D g(x,y) \, \mathrm{d}A$$

2. $$\iint_D cf(x,y) \, \mathrm{d}A = c \iint_D f(x,y) \, \mathrm{d}A$$

3. If $D = D_1 + D_2$, then

$$\iint_D f(x,y) \, \mathrm{d}A = \iint_{D_1} f(x,y) \, \mathrm{d}A + \iint_{D_2} f(x,y) \, \mathrm{d}A$$

4. If $f(x,y) \geq g(x,y)$, then
$$\iint_D f(x,y)\,\mathrm{d}A \geq \iint_D g(x,y)\,\mathrm{d}A$$
5. If $m \leq f(x,y) \leq M$ and $A(D)$ is the area of the region D, then
$$m \cdot A(D) \leq \iint_D f(x,y)\,\mathrm{d}A \leq M \cdot A(D).$$

Example 4.2.4. Estimate the integral $\iint_D e^{\sin x \cos y}\,\mathrm{d}A$, where D is a disk centered at origin with a radius of 2.

Answer.

Since $-1 \leq \sin x \leq 1 \quad$ and $\quad -1 \leq \cos y \leq 1$, we have
$$-1 \leq \sin x \cos y \leq 1.$$
Therefore,
$$e^{-1} \leq e^{\sin x \cos y} \leq e^{1}.$$
$$\iint_D e^{1}\,\mathrm{d}A \leq \iint_D e^{\sin x \cos y}\,\mathrm{d}A \leq \iint_D e^{1}\,\mathrm{d}A.$$
Recall that
$$\iint_D 1\,\mathrm{d}A = \text{Area of the disk} = 2^2\pi = 4\pi.$$
$$\iint_D e^{-1}\,\mathrm{d}A = e^{-1}\iint_D 1\,\mathrm{d}A = \frac{4\pi}{4} \qquad \text{and} \qquad \iint_D e^{1}\,\mathrm{d}A = 4e\pi.$$
$$\frac{4\pi}{e} \leq \iint_D e^{\sin x \cos y}\,\mathrm{d}A \leq 4e\pi.$$
□

4.3 Changing Variables in Double Integrals

Theorem 4.3.1 (Transformation of Double Integral).

$$\iint_R F(x,y)\,\mathrm{d}x\mathrm{d}y = \iint_{R'} F\Big(f(u,v), g(u,v)\Big)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}u\mathrm{d}v,$$

where $x = f(u,v)$ and $y = g(u,v)$. R' is the region in uv-plane which R is mapped under the trandofrmation $T = \begin{cases} x = f(u,v) \\ y = g(u,v) \end{cases}$.

Definition 4.3.1 (Jacobian). The Jacobian of transformation $T = \begin{cases} x = f(u,v) \\ y = g(u,v) \end{cases}$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \cdot \frac{\partial x}{\partial v}.$$

Example 4.3.1. If $u = x^2 - y^2$ and $v = 2xy$. Find $\frac{\partial(x,y)}{\partial(u,v)}$ in terms of u and v.

Answer.

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}} = \frac{1}{\begin{vmatrix} 2x & 2y \\ -2y & 2x \end{vmatrix}} = \frac{1}{4x^2 + 4y^2}$$

$$u = x^2 - y^2, \qquad v = 2xy$$

Note that:

$$\begin{aligned} \left(x^2 - y^2\right)^2 &= \left(x^2 + y^2\right)^2 - (2xy)^2 \\ u^2 &= \left(x^2 + y^2\right)^2 - v^2 \\ \left(x^2 + y^2\right)^2 &= u^2 + v^2 \\ x^2 + y^2 &= \pm\sqrt{u^2 + v^2} \end{aligned}$$

Therefore,

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\pm 4\sqrt{u^2 + v^2}}$$

□

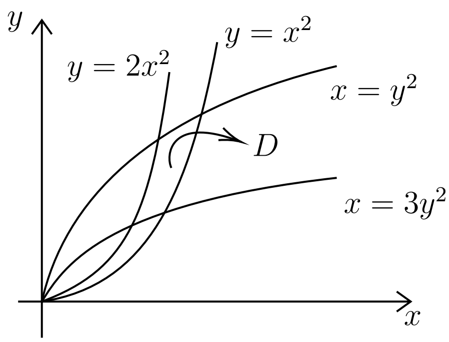
Theorem 4.3.2 (Absolute Value of Jacobian). In fact, the absolute value of Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ is the ratio between corresponding area elements in the xy-plane and the uv-plane.

$$\mathrm{d}A = \mathrm{d}x\mathrm{d}y = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v$$

Example 4.3.2. Find the area of the finite plane region bounded by the four parabolas:

$$y = x^2, \qquad y = 2x^2, \qquad x = y^2, \qquad x = 3y^2$$

Answer.



From $\begin{cases} y = x^2 \\ y = 2x^2 \end{cases}$, we know $\begin{cases} \frac{y}{x^2} = 1 \\ \frac{y}{x^2} = 2 \end{cases}$. Let $u = \dfrac{y}{x^2}$: $\begin{cases} u = 1 \\ u = 2 \end{cases}$.

Similarly, let $v = \dfrac{x}{y^2}$, then $\begin{cases} v = 1 \\ v = 3 \end{cases}$.

So, the region D is transformed to a rectangle in the uv-plane.

Let $u = \dfrac{y}{x^2}$ and $v = \dfrac{x}{y^2}$, where $1 \leq u \leq 2$ and $1 \leq v \leq 3$.

$$\iint_D \mathrm{d}A = \iint_R \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \mathrm{d}u\mathrm{d}v$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix}} = \frac{1}{\begin{vmatrix} -\dfrac{2y}{x^3} & \dfrac{1}{x^2} \\ \dfrac{1}{y^2} & -\dfrac{2x}{y^3} \end{vmatrix}} = \frac{1}{\dfrac{4}{x^2y^2} - \dfrac{1}{x^2y^2}} = \frac{x^2y^2}{3}.$$

Note that $uv = \dfrac{y}{x^2} \cdot \dfrac{x}{y^2} = \dfrac{1}{xy}$, so $u^2v^2 = \dfrac{1}{x^2y^2}$. Hence, $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{3u^2v^2}$.

Therefore,

$$\begin{aligned}
\iint_D \mathrm{d}A = \int_1^3 \int_1^2 \frac{1}{3u^2v^2} \,\mathrm{d}u\mathrm{d}v &= \frac{1}{3} \int_1^3 \int_1^2 \frac{1}{u^2v^2} \,\mathrm{d}u\mathrm{d}v \\
&= \frac{1}{3} \int_1^3 \left[-\frac{1}{uv^2} \right]_1^2 \mathrm{d}v \\
&= \frac{1}{3} \int_1^3 \left(-\frac{1}{2v^2} \right) \mathrm{d}v \\
&= -\frac{1}{6} \int_1^3 \frac{1}{v^2} \,\mathrm{d}v = -\frac{1}{6} \left[-\frac{1}{v} \right]_1^3 = -\frac{1}{6} \left(-1 + \frac{1}{3} \right) = \frac{1}{9}
\end{aligned}$$

□

4.4 Double Integral in Polar Coordinates

Theorem 4.4.1 (Double Integral in Polar Coordinates). In polar coordinates, $x^2+y^2 = r, \quad x = r\cos\theta, \quad y = r\sin\theta$. Therefore,

$$\iint_R F(x,y)\,\mathrm{d}A = \iint_R F(x,y)\,\mathrm{d}x\mathrm{d}y = \iint_{R'} F\Big(r\cos\theta, r\sin\theta\Big)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|\mathrm{d}r\mathrm{d}\theta.$$

Since

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & y_r \\ x_\theta & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r,$$

we have

$$\iint_R F(x,y)\,\mathrm{d}x\mathrm{d}y = \iint_{R'} F\Big(r\cos\theta, r\sin\theta\Big) r\,\mathrm{d}r\mathrm{d}\theta.$$

Example 4.4.1. Evaluate $\displaystyle\iint_D \frac{y^2}{x^2}\,\mathrm{d}A$, where D is the region limited to

$$0 \le a \le x^2+y^2 \le b \qquad y=0, \qquad x=y, \qquad x,y>0.$$

Answer.

$$\begin{aligned} I = \iint_D \frac{y^2}{x^2}\,\mathrm{d}A &= \int_0^{\pi/4}\int_a^b \tan^2\theta \cdot r\,\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{\pi/4}\left[\tan^2\theta\frac{r^2}{2}\right]_a^b \mathrm{d}\theta \\ &= \int_0^{\pi/4} \tan^2\theta\frac{b^2-a^2}{2}\,\mathrm{d}\theta \\ &= \frac{b^2-a^2}{2}\Big[\tan\theta - \theta\Big]_0^{\pi/4} \\ &= \frac{b^2-a^2}{2}\Big(1-\frac{\pi}{4}\Big). \end{aligned}$$

Remark. To evaluate $\displaystyle\int \tan^2\theta\,\mathrm{d}\theta = \int \frac{\sin^2\theta}{\cos^2\theta}\,\mathrm{d}\theta$, we apply $\sin^2\theta = 1-\cos^2\theta$:

$$\int \frac{\sin^2\theta}{\cos^2\theta}\,\mathrm{d}\theta = \int \frac{1-\cos^2\theta}{\cos^2\theta}\,\mathrm{d}\theta = \int \frac{1}{\cos^2\theta}\mathrm{d}\theta - \int \mathrm{d}\theta = \tan\theta - \theta + C.$$

□

Example 4.4.2. Show $I = \displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$.

Answer.

We try to find $I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right)^2 = \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x \cdot \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x$ Further, we have

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x \cdot \int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}\cdot e^{-y^2}\,\mathrm{d}x\mathrm{d}y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,\mathrm{d}x\mathrm{d}y$$

Then, we change it to the polar coordinate:

$$\begin{aligned} I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,\mathrm{d}r\mathrm{d}\theta &= 2\pi \int_0^{\infty} e^{-r^2} r\,\mathrm{d}r \\ &= 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} \\ &= -\pi\left(\lim_{t\to\infty}\frac{1}{e^{t^2}} - e^0\right) \\ &= \pi(0-1) = \pi. \end{aligned}$$

□

4.5 Triple Integrals

Definition 4.5.1 (Triple Integral). Find a bounded function $f(x, y, z)$ defined on a rectangular box, $B : \begin{cases} x_1 \le x \le x_2 \\ y_1 \le y \le y_2 \\ z_1 \le z \le z_2 \end{cases}$, then, the triple integral on that box in defined as

$$\iiint_B f(x,y,z)\,\mathrm{d}V = \lim_{n,m,l\to\infty} \sum_{k=1}^{n}\sum_{j=1}^{m}\sum_{i=1}^{l} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\underbrace{\Delta x\Delta y\Delta z}_{\Delta V}$$

if the limit exists.

Theorem 4.5.1 (Fubini's Theorem for Triple Integral). If $f(x, yz)$ is continuous over a box B, where B is defined by $B = \{(x, y, z) \mid x_1 \le x \le x_2, y_1 \le y \le y_2, z_1 \le z \le z_2\}$, then

$$\iiint_B f(x,y,z)\,\mathrm{d}V = \int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Theorem 4.5.2.

$$\iiint_B \mathrm{d}V = V(B) = \text{Volume of the box } B$$

In more general cases,

$$\iiint_E \mathrm{d}V = V(E) = \text{Volume of a more general bounded region } E,$$

where E is a general bounded region.

Theorem 4.5.3 (Volume of a Sphere).

$$V(\text{Sphere}) \iiint_E \mathrm{d}V = \frac{4}{3}\pi a^3, \text{ where } E \text{ is bounded by } x^2+y^2+z^2 \leq a$$

Example 4.5.1. Evaluate $\iiint_E 2 + x - \sin z \,\mathrm{d}V$, where E is bounded by $x^2+y^2+z^2 \leq a$

Answer.

x and $\sin z$ are odd functions, so integrals of them are 0 on a symmetric region.

Note that E, by definition, is sphere centered at origion, with a radius of a, which is a symmetric region, so we have

$$\iiint_E x \,\mathrm{d}V = \iiint_E \sin z \,\mathrm{d}V = 0.$$

Plugging into the integral, we will have

$$\iiint_E 2 + x - \sin z \,\mathrm{d}V = \iiint_E 2 \,\mathrm{d}V + \iiint_E x \,\mathrm{d}V + \iiint_E \sin z \,\mathrm{d}V = \iiint_E 2 \,\mathrm{d}V = \frac{8}{3}\pi a^3.$$

□

Example 4.5.2. Evaluate $\iiint_B xyz^2 \,\mathrm{d}V$, where $B = \{(x,y,z) \mid 0 \leq x \leq 1, -1 \leq y \leq 2, 0 \leq z \leq 3\}$.

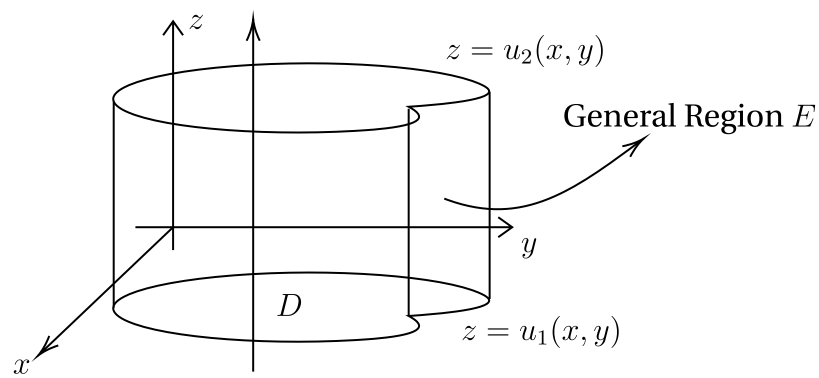
Answer.

$$\begin{aligned}
\iiint_B xyz^2 \,\mathrm{d}V &= \int_0^3 \int_{-1}^2 \int_0^1 xyz^2 \,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \int_0^3 \int_{-1}^2 \left[\frac{1}{2}x^2yz^2\right]_0^1 \mathrm{d}y\mathrm{d}z \\
&= \int_0^3 \int_{-1}^2 \frac{1}{2}yz^2 \,\mathrm{d}y\mathrm{d}z \\
&= \int_0^3 \left[\frac{1}{4}y^2z^2\right]_{-1}^2 \mathrm{d}z \\
&= \int_0^3 \frac{1}{4}(4)z^2 - \frac{1}{4}z^2 \,\mathrm{d}z \\
&= \left[\frac{1}{3}z^3 - \frac{1}{12}z^3\right]_0^3 \\
&= \frac{1}{3}(27) - \frac{1}{12}(27) = 9 - \frac{9}{4} = \frac{27}{4}
\end{aligned}$$

□

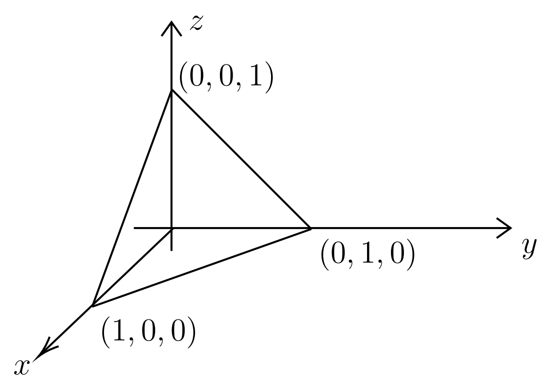
Theorem 4.5.4 (Triple Integral Over a General Region). If we can write $z = u(x, y)$ as function of x and y, then we can change the triple integral into double integral. The following diagram shows this case.

$$\begin{aligned}\iiint_E f(x,y,z)\,\mathrm{d}V &= \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,\mathrm{d}z\right]\mathrm{d}A\\ &= \int_a^b \int_{g_1}^{g_2} \int_{u_1}^{u_2} f(x,y,z)\,\mathrm{d}z\mathrm{d}x\mathrm{d}y, \quad g(y) = x\\ \text{OR} \quad &= \int_c^d \int_{h_1}^{h_2} \int_{u_1}^{u_2} f(x,y,z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x, \quad h(x) = y\end{aligned}$$



Example 4.5.3. Evaluate $\iiint_E z\,\mathrm{d}V$, where E is the solid tetrahedron bounded by the following planes:

$$x = 0; \qquad y = 0; \qquad z = 0; \qquad x + y + z = 1.$$



Answer.

$$\begin{aligned}
\iiint_E z \, \mathrm{d}V &= \iint_D \left[\int_0^{1-x-y} z \, \mathrm{d}z\right] \mathrm{d}A \\
&= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, \mathrm{d}z\mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \int_0^{1-x} \left[d\frac{1}{2}z^2\right]_0^{1-x-y} \mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \int_0^{1-x} \frac{1}{2}(1-x-y)^2 \, \mathrm{d}y\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 \int_0^{1-x} 1 + x^2 + y^2 - 2x - 2y + 2xy \, \mathrm{d}y\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 \left[y + x^2y + \frac{1}{3}y^3 - 2xy - \frac{2}{2}y^2 + \frac{2}{2}xy^2\right]_0^{1-x} \mathrm{d}x \\
&= \frac{1}{2}\int_0^1 (1-x) + x^2(1-x) + \frac{1}{3}(1-x)^3 - 2x(1-x) - (1-x)^2 + x(1-x)^2 \, \mathrm{d}x \\
&= \frac{1}{2}\int_0^1 \left(1 - x + x^2 - x^3 + \frac{1}{3}(1-x)^3 - 2x + 2x^2 - 1 + 2x - x^2 + x - 2x^2 + x^3\right) \mathrm{d}x \\
&= \frac{1}{2}\int_0^1 \frac{1}{3}(1 - x^3 + 3x^2 - 3x) \, \mathrm{d}x \\
&= \frac{1}{6}\left[x - \frac{1}{4}x^4 + \frac{3}{3}x^3 - \frac{3}{2}x^2\right]_0^1 = \frac{1}{6}\left(1 - \frac{1}{4} + 1 - \frac{3}{2}\right) = \frac{1}{6}\left(\frac{1}{4}\right) = \frac{1}{24}.
\end{aligned}$$

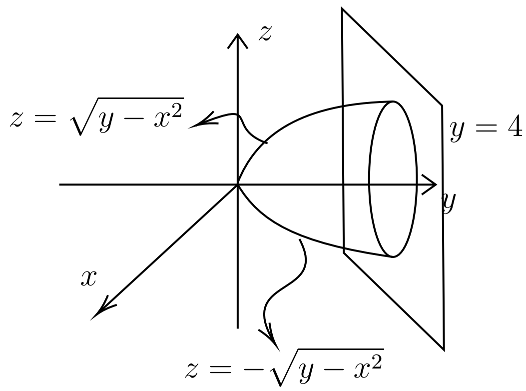
□

Extension. Similarly, we can have other types of triple integrals over the general region:

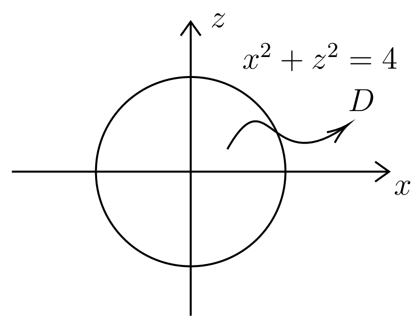
$$\iiint_E f(x,y,z) \, \mathrm{d}V = \iint_D \left[\int_{u_1(y,z)}^{u_1(y,z)} f(x,y,z) \, \mathrm{d}x\right] \mathrm{d}A$$

$$\iiint_E f(x,y,z) \, \mathrm{d}V = \iint_D \left[\int_{u_1(x,z)}^{u_1(x,z)} f(x,y,z) \, \mathrm{d}y\right] \mathrm{d}A$$

Example 4.5.4. Evaluate $\iiint_E \sqrt{x^2+z^2} \, \mathrm{d}V$, where E is the region bounded by $y = x^2 + z^2$ and $y = 4$.



Answer.



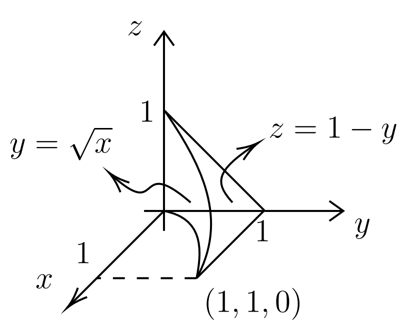
$$\iiint_E \sqrt{x^2+z^2}\,\mathrm{d}V = \iint_D \left[\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,\mathrm{d}y\right]\mathrm{d}A$$
$$= \iint_D \left[(4 - x^2 - z^2)\sqrt{x^2+z^2}\right]\mathrm{d}A$$

Now, change to polar coordinate: $r^2 = x^2 + z^2, \quad 0 \le r \le 2, \quad 0 \le \theta \le 2\pi$. So,

$$\iiint_E \sqrt{x^2+z^2}\,\mathrm{d}V = \iint_{D'} (4 - r^2)\sqrt{r^2} \cdot r\,\mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi}\int_0^2 4r^2 - r^4\,\mathrm{d}r\mathrm{d}\theta$$
$$= 2\pi\left[\frac{4}{3}r^3 - \frac{1}{5}r^5\right]_0^2$$
$$= 2\pi\left(\frac{4}{3}(8) - \frac{32}{5}\right) = \frac{128}{15}\pi$$

□

Example 4.5.5. Given $\displaystyle\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x$. Rewrite the triple integral using other five orders.



Answer.

① Change to $\mathrm{d}z\mathrm{d}x\mathrm{d}y$:

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \int_0^1 \int_0^{y^2} \int_0^{1-y} f(x,y,z)\,\mathrm{d}z\mathrm{d}x\mathrm{d}y$$

② Change to $\mathrm{d}x\mathrm{d}y\mathrm{d}z$:

$$\begin{aligned}\iiint_E f(x,y,z)\,\mathrm{d}V &= \iint_D \left[\int_0^{y^2} f(x,y,z)\,\mathrm{d}x\right]\mathrm{d}A \\ &= \int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z)\,\mathrm{d}x\mathrm{d}z\mathrm{d}y\end{aligned}$$

③ Change to $\mathrm{d}x\mathrm{d}y\mathrm{d}z$: From $z = 1-y$, we have $y = 1-z$. So,

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

④ Change to $\mathrm{d}y\mathrm{d}z\mathrm{d}x$:

$$\begin{aligned}\iiint_E f(x,y,z)\,\mathrm{d}V &= \iint_D \left[\int_{\sqrt{x}}^{1-z} f(x,y,z)\,\mathrm{d}y\right]\mathrm{d}A \\ &= \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,\mathrm{d}y\mathrm{d}z\mathrm{d}x\end{aligned}$$

⑤ Change to $\mathrm{d}y\mathrm{d}x\mathrm{d}z$: Since $z = 1-\sqrt{x}$, we have $\sqrt{x} = 1-z$, or $x = (1-z)^2$:

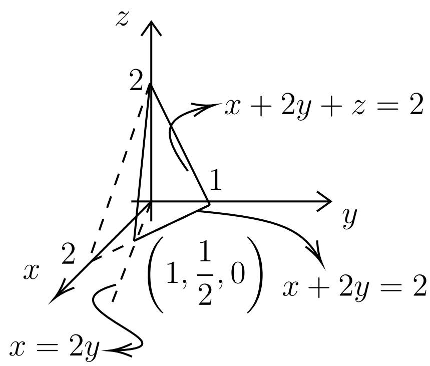
$$\iiint_E f(x,y,z)\,\mathrm{d}V = \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,\mathrm{d}y\mathrm{d}x\mathrm{d}z$$

□

Remark. One application of triple integral is to find volume of a region.

Example 4.5.6. Find the volume of the region bounded by the following planes:

$$x + 2y + z = 2, \qquad x = 2y, \qquad x = 0, \qquad z = 0.$$



Answer.

From $x + 2y + x = 2$, we know that $z = 2 - x - 2y$. So we have

$$
\begin{aligned}
V = \iiint_E 1 \, \mathrm{d}V &= \iint_D \left[\int_0^{2-x-2y} 1 \, \mathrm{d}z\right] \mathrm{d}A \\
&= \int_0^1 \int_{x/2}^{(2-x)/2} \int_0^{2-x-2y} 1 \, \mathrm{d}z\mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \int_{x/2}^{(2-x)/2} (2 - x - 2y) \, \mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \left[(2-x)y - y^2\right]_{x/2}^{(2-x)/2} \mathrm{d}x \\
&= \int_0^1 \left((2-x)(1-x) - \frac{1}{4}x^2 - 1 + x + \frac{1}{4}x^2\right) \mathrm{d}x \\
&= \int_0^1 (x^2 - 2x + 1) \, \mathrm{d}x \\
&= \left[\frac{1}{3}x^3 - x^2 + x\right]_0^1 = \frac{1}{3} - 1 + 1 = \frac{1}{3}
\end{aligned}
$$

□

4.6 Changing Variables in Triple Integrals

Theorem 4.6.1 (Change of Variables in Triple Integrals). Consider the transformation $T = \begin{cases} x = f(u,v,w) \\ y = g(u,v,w) \\ z = h(u,v,w) \end{cases}$. We have $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z = \left|\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\right|\mathrm{d}u\mathrm{d}v\mathrm{d}w$, where

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \\ \dfrac{\partial x}{\partial w} & \dfrac{\partial y}{\partial w} & \dfrac{\partial z}{\partial w} \end{vmatrix}.$$

Then, we have

$$\iiint_E f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{E'} g(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\mathrm{d}u\mathrm{d}v\mathrm{d}w.$$

Remark. The determinant of triangular and diagonal matrices is the product of the elements on the main diagonal. Suppose matrix **A** and **B** are defined as follows:

$$\mathbf{A} = \begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

Then $\det(\mathbf{A}) = \det(\mathbf{B}) = abc.$

Example 4.6.1. Find the volume of ellipsoid is given by $E : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \leq 1$

Answer.

Consider the transformation: $x = au, \quad y = bv, \quad z = cw.$

Then,

$$E' : \frac{(au)^2}{a^2} + \frac{(bv)^2}{b^2} + \frac{(cw)^2}{c^2} \leq 1$$
$$u^2 + v^2 + w^2 \leq 1$$

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

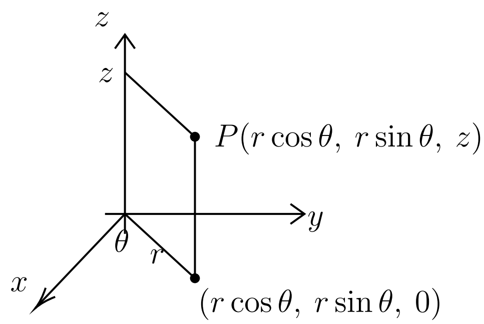
So,

$$\iiint_E 1 \, \mathrm{d}V = \iiint_{E'} abc \, \mathrm{d}V = abc \times V(\text{ball with radius} = 1) = abc\left(\frac{4}{3}\pi\right).$$

□

Remark. In 3D, there are two alternatives to Cartesian coordinate system: Cylindrical coordinate system and spherical coordinate system.

Definition 4.6.1 (Cylindrical Coordinate System). Uses polar coordinate in the xy-plane while retaining the Cartesian z coordinate for measuring vertical distance.



In Cylindrical Coordinate system, $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. So,

$$\left|\frac{\partial(x, y, z)}{\partial(r, \theta, z}\right| = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 0 \end{vmatrix} = r.$$

So,

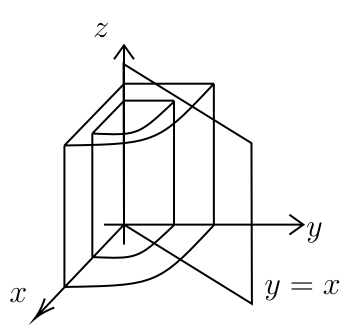
$$\mathrm{d}V = r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z.$$

Theorem 4.6.2 (Change Triple Integrals to Cylindrical Coordinate System).

$$\iiint_E f(x, y, z) \, \mathrm{d}V = \int_{z=u_1(x,y)}^{z=u_2(x,y)} \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r\cos\theta, r\sin\theta, z) r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z.$$

Example 4.6.2. Evaluate $I = \iiint_E x^2 + y^2 \, \mathrm{d}V$ over the first octant region bounded by the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and planes $z = 0,\ z = 1,\ x = 0$, and $y = x$.

Answer.



Change to Cylindrical Coordinate System: $r^2 = x^2 + y^2$., where $1 \le r \le 2$, $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$, $0 \le z \le 1$. Then,

$$\begin{aligned} I &= \int_0^1 \int_{\pi/4}^{\pi/2} \int_1^2 r^2 \cdot r \, \mathrm{d}r\mathrm{d}\theta\mathrm{d}z \\ &= (1-0)\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\left(\frac{2^4}{4} - \frac{1^4}{4}\right) = \frac{15}{16}\pi \end{aligned}$$

□

Example 4.6.3. Evaluate $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2) \, \mathrm{d}z\mathrm{d}y\mathrm{d}x$.

Answer.

Change to Cylindrical Coordinate system: $r^2 = x^2 + y^2$. So, $r \le z \le 2$.

Since $-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$, so $0 \le y^2 \le 4 - x^2$

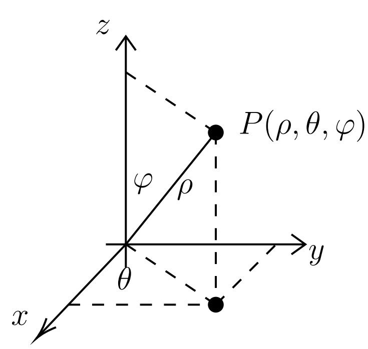
That is, $0 \le y^2 + x^2 \le 4$, or $0 \le r^2 \le 4$.

So, $0 \le r \le 2$.

$$\begin{aligned} \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2) \, \mathrm{d}z\mathrm{d}y\mathrm{d}x &= \int_0^{\pi} \int_0^2 \int_r^2 r^2 \cdot r \, \mathrm{d}z\mathrm{d}r\mathrm{d}\theta \\ &= (2\pi) \int_0^2 r^3(2-r) \, \mathrm{d}r \\ &= (2\pi) \int_0^2 2r^3 - r^4 \, \mathrm{d}r \\ &= (2\pi) \left[\frac{1}{2}r^4 - \frac{1}{5}r^5\right]_0^2 \\ &= (2\pi)\left(8 - \frac{32}{5}\right) = \frac{16}{5}\pi \end{aligned}$$

□

Definition 4.6.2 (Spherical Coordinate System). Here we define ρ is the distance from the origin to P, φ is the angle between the line OP and the positive z-axis ($0 \le \varphi \le \pi$), and θ is the angle between OP' (the projection of OP onto the xy-plane) and the positive x-axis ($0 \le \theta \le 2\pi$). So a point $P(\rho, \theta, \varphi)$ is represented in the following graph.



Using trigonometric identities, we know $z = \rho\cos(\varphi)$ and $OP' = \rho\sin(\varphi)$. Then, $x = \rho\sin(\varphi)cos(\theta)$ and $y = \rho\sin(\varphi)\sin(\theta)$. Also, applying the formula, we know $\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \rho^2\sin(\varphi)$. Therefore,

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \int_c^d\int_\alpha^\beta\int_a^b f\Big(\rho\sin(\varphi)\cos(\theta),\ \rho\sin(\varphi)\sin(\theta),\ \rho\cos(\varphi)\Big)\rho^2\sin(\varphi)\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\varphi,$$

where $a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \varphi \le d.$

Example 4.6.4. Evaluate $\displaystyle\iiint_E e^{(x^2+y^2+z^2)^{3/2}}\,\mathrm{d}V$, where $E = \{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$.

Answer.

Change to spherical coordinate: $\rho^2 = x^2+y^2+z^2$.

$$\begin{aligned}\iiint_E e^{(x^2+y^2+z^2)^{3/2}}\,\mathrm{d}V &= \iiint_{E'} e^{(\rho^2)^{3/2}}\rho^2\sin(\varphi)\,\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\varphi\\ &= \iiint_{E'} e^{\rho^3}\rho^2\sin(\varphi)\,\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\varphi\\ &= \int_0^\pi\int_0^{2\pi}\int_0^1 \rho^2 e^{\rho^3}\sin(\varphi)\,\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\varphi\\ &= \int_0^\pi \sin(\varphi)\,\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\theta\int_0^1\rho^2e^{\rho^3}\,\mathrm{d}\rho.\end{aligned}$$

Let $u = \rho^3$, then $\mathrm{d}u = 3\rho^2\mathrm{d}\rho$. So, $\displaystyle\int \rho^2e^{\rho^3}\,\mathrm{d}\rho = \frac{1}{3}\int e^u\,\mathrm{d}u = \frac{1}{3}e^u = \frac{1}{3}e^{\rho^3}$.

So,

$$\begin{aligned}\iiint_E e^{(x^2+y^2+z^2)^{3/2}}\,\mathrm{d}V &= \Big[-\cos(\varphi)\Big]_0^\pi(2\pi)\left[\frac{1}{3}e^{\rho^3}\right]_0^1\\ &= (1+1)(2\pi)\left(\frac{1}{3}e - \frac{1}{3}\right)\\ &= \frac{4}{3}\pi(e-1).\end{aligned}$$

□

4.7 Applications of Multiple Integrals

Theorem 4.7.1 (Surface Area). The key idea is to use the tangent plane at any point like $P_{ij}(x_i, y_j, z_k)$ to approximate the surface near the point P_{ij}.

Divide region D into small rectangles, R_{ij}. So,

$$\Delta A = A(R_{ij}) = \Delta x\Delta y$$

Let (x_i, y_j) be a point on R_{ij}, and its corresponding point on the surface is given by

$$P_{ij}(x_i, y_j, f(x_i, y_j))$$

The tangent plane to the surface S at point P_{ij} is an approximation of the surface around P_{ij}. Therefore, $\Delta S_{ij} \approx \Delta T_{ij}$. So,

$$A(S) \approx \sum_{i=1}^{n} \sum_{j=1}^{m} \Delta T_{ij}$$

and

$$A(S) = \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} \Delta T_{ij}$$

To find ΔT_{ij}, we use cross product: $A(\Delta T_{ij}) = \left|\vec{\mathbf{a}} \times \vec{\mathbf{b}}\right|$.

- Slope of $\vec{\mathbf{a}} = f_x(x_i, y_j) = \dfrac{\Delta z}{\Delta x}$

$$\implies \Delta z = \Delta x f_x(x_i, y_j), \quad \vec{\mathbf{a}} = \Delta x \hat{\mathbf{i}} + \Delta x f_x(x_i, y_j)\hat{\mathbf{k}}.$$

- Slope of $\vec{\mathbf{b}} = f_y(x_i, y_j) = \dfrac{\Delta z}{\Delta y}$

$$\implies \Delta z = \Delta y f_y(x_i, y_j), \quad \vec{\mathbf{b}} = \Delta y \hat{\mathbf{j}} + \Delta y f_y(x_i, y_j)\hat{\mathbf{k}}.$$

So,

$$\begin{aligned}
\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \Delta x & 0 & \Delta x f_x(x_i, y_j) \\ 0 & \Delta y & \Delta y f_y(x_i, y_j) \end{vmatrix} &= (-f_x(x_i, y_j)\hat{\mathbf{i}} - f_y(x_i, y_j)\hat{\mathbf{j}} + \hat{\mathbf{k}})\Delta x \Delta y \\
&= (-f_x(x_i, y_j)\hat{\mathbf{i}} - f_y(x_i, y_j)\hat{\mathbf{j}} + \hat{\mathbf{k}})\Delta A
\end{aligned}$$

So,

$$\begin{aligned}
A(\Delta T_{ij}) &= \left|\vec{\mathbf{a}} \times \vec{\mathbf{b}}\right| \\
&= \sqrt{f_x^2(x_i, y_j) + f_y^2(x_i, y_j) + 1}\, \Delta A
\end{aligned}$$

Therefore,

$$\begin{aligned}
S &= \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} \Delta T_{ij} \\
&= \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} \sqrt{f_x^2(x_i,y_j)+f_y^2(x_i,y_j)+1}\,\Delta A \\
&= \boxed{\iint_D \sqrt{f_x^2(x,y)+f_y^2(x,y)+1}\,\mathrm{d}A} \\
&= \boxed{\iint_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}\,\mathrm{d}A}
\end{aligned}$$

Example 4.7.1. Find the surface area of the paraboloid $z = x^2 + y^2$ that lies under $z = 9$.

Answer.

$$\begin{aligned}
S &= \iint_D \sqrt{1+(2x)^2+(2y)^2}\,\mathrm{d}A \\
&= \iint_D \sqrt{1+4x^2+4y^2}\,\mathrm{d}A
\end{aligned}$$

Change to polar coordinate: $0 \le r \le 3$ and $0 \le \theta \le 2\pi$:

$$\begin{aligned}
S &= \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}r\mathrm{d}r\mathrm{d}\theta \\
&= 2\pi \int_0^3 r\cdot\sqrt{1+4r^2}\,\mathrm{d}r
\end{aligned}$$

Let $u = 1 + 4r^2$, so $\mathrm{d}u = 8r\ \mathrm{d}r$. So,

$$\begin{aligned}
\int r\sqrt{1+4r^2}\,\mathrm{d}r &= \frac{1}{8}\int \sqrt{u}\,\mathrm{d}u \\
&= \frac{1}{8}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{12}u^{3/2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
S &= 2\pi \int_0^3 r\cdot\sqrt{1+4r^2}\,\mathrm{d}r \\
&= 2\pi\left[\frac{1}{12}\left(1+4r^2\right)^{3/2}\right]_0^3 \\
&= \frac{\pi}{6}(37\sqrt{37}-1).
\end{aligned}$$

□

Example 4.7.2. Find the area of the part of the plane $z = ax+by+c$ that projects onto a region in the xy-plane with an area of A.

Answer.

$$\text{Area} = \iint_D \sqrt{a^2+b^2+1}\,\mathrm{d}A = \sqrt{a^2+b^2+1}\iint_D \mathrm{d}A$$

Since $\iint_D \mathrm{d}A = A$ is given,

$$\text{Area} = \sqrt{a^2+b^2+1}(A) = A\sqrt{a^2+b^2+1}.$$

□

Definition 4.7.1 (Mass from Density Function). Let D be a lamina (a thin plate) made of materials whose density varies across D. Let $\rho(x,y)$ be the density of D at point (x,y), we define

$$m(D) = \iint_D \rho(x,y)\,\mathrm{d}A$$

as the total mass of D with density function ρ.

Remark. If we change $\rho(x,y)$ to be probability functions, $m(D)$ can be regarded as the cumulative probability.

Definition 4.7.2 (Center of Mass). The center of mass is denoted by the point $(\bar{x},\bar{y})$ on D such that if we place a support at that point, the lamina D will have a perfect balance.

Definition 4.7.3 (Moment). We define the moment of the lamina D over the y-axis as

$$\iint_D x\rho(x,y)\,\mathrm{d}A$$

and the moment of the lamina D over the x-axis as

$$\iint_D y\rho(x,y)\,\mathrm{d}A.$$

Theorem 4.7.2 (Calculate Center of Mass). We use moment of the lamina to calculate the center of mass:

$$\bar{x} = \frac{\iint_D x\rho(x,y)\,\mathrm{d}A}{m(D)}; \qquad \bar{y} = \frac{\iint_D y\rho(x,y)\,\mathrm{d}A}{m(D)}.$$

Example 4.7.3. The geometric model of a material body is a plane region R bounded by $y = x^2$ and $y = \sqrt{2-x^2}$ on the interval $[0,1]$. The density function is $\rho(x,y) = xy$. Find the center of mass of R.

Answer.

We know

$$m(D) = \iint_D xy\,\mathrm{d}A = \int_0^1 \int_{x^2}^{\sqrt{2-x^2}} xy\,\mathrm{d}y\mathrm{d}x = \frac{7}{24}.$$

Applying the formula to calculate the center of mass, we get

$$\bar{x} = \frac{\iint_D x\rho(x,y)\,\mathrm{d}A}{m(D)} = \frac{\frac{17}{105}}{\frac{7}{24}}$$

and

$$\bar{y} = \frac{\iint_D y\rho(x,y)\,\mathrm{d}A}{m(D)} = \frac{\frac{13}{120} + \frac{4\sqrt{2}}{15}}{\frac{7}{24}}.$$

□

4.8 Multiple Integral – Practice

Example 4.8.1. If D is the triangle with vertices $(-2,0),\ (0,4)$, and $(8,0)$, calculate $\iint_D xy^2\,\mathrm{d}A$.

Answer.

- Using the order $\mathrm{d}y\mathrm{d}x$, we have

$$\int_{-2}^0 \int_0^{2x+4} xy^2\,\mathrm{d}y\mathrm{d}x + \int_0^8 \int_0^{-x/2+4} xy^2\,\mathrm{d}y\mathrm{d}x$$

  It is not easy to calculate the integral as two parts.

- Using the order $\mathrm{d}x\mathrm{d}y$, we have

$$\begin{aligned}
\int_0^4 \int_{-2+y/2}^{8-2y} xy^2\,\mathrm{d}x\mathrm{d}y &= \int_0^4 \left[\frac{1}{2}x^2y^2\right]_{-2+y/2}^{8-2y} \mathrm{d}y \\
&= \int_0^4 30y^2 - 15y^3 + \frac{15}{8}y^4\,\mathrm{d}y \\
&= \left[30y^2 - 15y^3 + \frac{15}{8}y^4\right]_0^4 \\
&= 640 - 960 + 384 = 64.
\end{aligned}$$

□

Example 4.8.2. If D is the region bounded by $y = x^2$ and $y = 8 - x^2$, calculate $\iint_D x^3\,\mathrm{d}A$.

Answer.

D is a symmetric region about $x = 0$ and function $f(x, y) = x^3$ is an odd function with respect to x. Therefore,

$$\iint_D x^3 \, \mathrm{d}A = 0.$$

□

Example 4.8.3. Calculate the area of the region bounded by two parabolas $y = x^2$ and $x = y^2$.

Answer.

$$\begin{aligned} A(D) = \iint_D 1 \, \mathrm{d}A &= \int_0^1 \int_{x^2}^{\sqrt{x}} 1 \, \mathrm{d}y\mathrm{d}x \\ &= \int_0^1 \sqrt{x} - x^2 \, \mathrm{d}x \\ &= \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x\right]_0^1 \\ &= \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \end{aligned}$$

□

Example 4.8.4. Let D be the unit disk: $x^2 + y^2 \leq 1$. Calculate $\iint_D (2 - x)(3 + y) \, \mathrm{d}A$.

Answer.

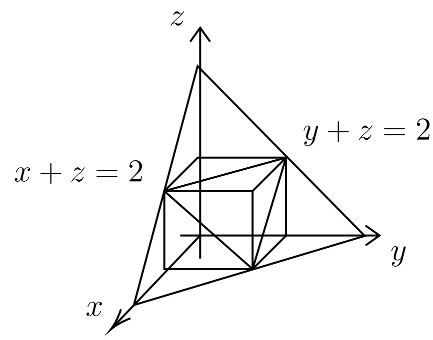
D is a symmetric region in x and y. So,

$$\begin{aligned} \iint_D (2 - x)(3 + y) \, \mathrm{d}A &= \iint_D 6 - 3x + 2y - xy \, \mathrm{d}A \\ &= \iint_D 6 \, \mathrm{d}A - \underbrace{\iint_D 3x \, \mathrm{d}A}_{=0 \text{ (symmetric in } x)} + \underbrace{\iint_D -xy + 2y \, \mathrm{d}A}_{=0 \text{ (symmetric in } y)} \\ &= 6 \times A(D) = 6\pi. \end{aligned}$$

□

Example 4.8.5. Find $\iiint_E x \, \mathrm{d}V$, where E is the tetrahedron bounded by the plane

$$x = 1, \quad y = 1, \quad z = 1, \quad x + y + z = 2.$$



Answer.

$$\begin{aligned}\iiint_E x \, \mathrm{d}V &= \iint_D \left[\int_{2-x-y}^{1} x \, \mathrm{d}z\right] \mathrm{d}A \\ &= \int_0^1 \int_{1-x}^{1} \int_{2-x-y}^{1} x \, \mathrm{d}z\mathrm{d}y\mathrm{d}x \\ &= \int_0^1 \int_{1-x}^{1} x(1-2+x+y) \, \mathrm{d}y\mathrm{d}x \\ &= \int_0^1 \int_{1-x}^{1} x^2 + xy - x \, \mathrm{d}y\mathrm{d}x \\ &= \int_0^1 x^3 + x^2 - \frac{1}{2}x^3 - x^2 \, \mathrm{d}x \\ &= \int_0^1 \frac{1}{2}x^3 \, \mathrm{d}x = \left[\frac{1}{2}x^3\right]_0^1 = \frac{1}{8}.\end{aligned}$$

□

Example 4.8.6. Plot the cylindrical coordinate of $\left(4, \frac{\pi}{3}, -3\right)$ and find its rectangular coordinates.

Answer.

$$r = 4, \quad \theta = \frac{\pi}{3}, \quad z = -3.$$

$$x = r\cos\theta = 3 \cdot \cos\left(\frac{\pi}{3}\right) = 4 \cdot \frac{1}{2} = 2$$

$$y = r\sin\theta = 3 \cdot \sin\left(\frac{\pi}{3}\right) = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}.$$

Rectangular coordinate: $(2, 2\sqrt{3}, -3)$.

□

Example 4.8.7. Find the volume enclosed by the cone $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 + z^2 = 2$.

Answer.

Change to cylindrical coordinate: $x^2 + y^2 = r^2$ and $z = z$:

$$0 \leq r \leq \sqrt{2}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 1.$$

So,

$$\iiint_E \mathrm{d}V = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_0^1 \mathrm{d}z\mathrm{d}r\mathrm{d}\theta = 2\pi(\sqrt{2})(1) = 2\sqrt{2}\pi.$$

□

5 Vector Calculus

5.1 Vector Fields

Definition 5.1.1 (Vector Field). Let D be a region (or a set) in $\mathbb{R}^n$. A vector field on $\mathbb{R}^n$ is a function $\vec{\mathbf{F}}$ that assigns to each point $(x_1, \cdots, x_n)$ a n-dimensional vector $\vec{\mathbf{F}}(x_1, \cdots, x_n)$.

Example 5.1.1.

$$\vec{\mathbf{F}}(x, y) = P(x, y)\hat{\mathbf{i}} + Q(x, y)\hat{\mathbf{j}},$$

where P and Q are scalar functions. Sometimes, P and Q are called scalar fields.

$$\vec{\mathbf{F}}(x, y, z) = P(x, y, z)\hat{\mathbf{i}} + Q(x, y, z)\hat{\mathbf{j}} + R(x, y, z)\hat{\mathbf{k}},$$

where P, Q, and R are scalar functions or scalar fields.

Remark. In fact, vector fields can model velocity, magnetic force, fluid motion, and gradient.

Definition 5.1.2 (Gradient Fields). let f be a scalar function of two (or three) variables on $\mathbb{R}^2$ (or $\mathbb{R}^3$). Its gradient is a vector field on $\mathbb{R}^2$ (or $\mathbb{R}^3$) given by

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}}$$

or

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}.$$

Example 5.1.2. Find the gradient vector field of $f(x, y) = x^2y - y^3$.

Answer.

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} = 2xy\hat{\mathbf{i}} + (x^2 - 3y^2)\hat{\mathbf{j}}$$

□

Remark. Properties of Gradient Fields

- Gradient vectors are perpendicular to the level curves
- Gradient vectors point in the direction of maximum change in value of the function at a given point.
- The magnitudes of gradient vectors are a measure of local intensity change at a given point.

5.2 Line Integrals

In this section, we define line integral similar to a single integral, but instead of interval, we integrate over a curve.

Definition 5.2.1 (Line Integral). Let f be defined on a differentiable curve C, where

$$C = \begin{cases} x(t) \\ y(t) \end{cases}, \quad a \leq t \leq b.$$

We choose (x_i^*, y_i^*) on sub-arc correspond to t_i^*. We calculate

$$\sum_{i=1}^{n} f(x_i^*, y_i*)\Delta S_i.$$

When $n \to \infty$, we define the line integral of f along curve C as

$$\int_C f(x,y)\,\mathrm{d}s = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta S_i$$

if the limit exists.

Theorem 5.2.1 (Length of a Curve). The length of a curve C defined by $\begin{cases} x(t) \\ y(t) \end{cases}$ is given by

$$L = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

Theorem 5.2.2 (Calculating Line Integrals). Applying Theorem 5.2.1, we have

$$\int_C f(x,y)\,\mathrm{d}s = \int_a^b f(x,y)\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

Example 5.2.1. Evaluate $\displaystyle\int_C 2 + x^2y\,\mathrm{d}s$ over the upper half of the unit circle $x^2 + y^2 = 1$.

Answer.

We know $C : \begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases}$, $0 \le t \le \pi$. So, $x'(t) = -\sin t$ and $y'(t) = \cos t$.

$$\begin{aligned} \int_C 2 + x^2 y \, \mathrm{d}s &= \int_0^\pi (2 + x^2 y)\sqrt{(-\sin t)^2 + (\cos t)^2} \, \mathrm{d}t \\ &= \int_0^\pi (2 + \cos^2 t \sin^t) \, \mathrm{d}t \\ &= \Big[2t\Big]_0^\pi - \frac{1}{3}\Big[\cos^3 t\Big]_0^\pi = 2\pi - \frac{1}{2}(-2) = 2\pi + \frac{2}{3}. \end{aligned}$$

□

Theorem 5.2.3 (Price-weise Smooth Line Integrals). If C is a piece-wise smooth curve defined by $C_1 + C_2 + \cdots + C_n$. Then, the line integral over C is

$$\int_C f(x,y) \, \mathrm{d}x = \int_{C_1} f(x,y) \, \mathrm{d}s + \int_{C_2} f(x,y) \, \mathrm{d}s + \cdots + \int_{C_n} f(x,y) \, \mathrm{d}s$$

Theorem 5.2.4 (Vector Representation of a Line Segment). The vector representation of a line segment starts at $\vec{\mathbf{r}}_0$ and ends at $\vec{\mathbf{r}}_1$ is given by

$$\vec{\mathbf{r}}(t) = (1-t)\vec{\mathbf{r}}_0 + t\vec{\mathbf{r}}_1 \qquad 0 \le t \le 1.$$

Definition 5.2.2 (Line Integrals with Respect to x and y).

$$\int_C f(x,y) \, \mathrm{d}x = \lim_{n \to \infty} \sum_{i=1}^n f(x_i^*, y_i^*)\Delta x_i = \int_a^b f(x(t), y(t))x'(t) \, \mathrm{d}t$$

$$\int_C f(x,y) \, \mathrm{d}y = \lim_{n \to \infty} \sum_{i=1}^n f(x_i^*, y_i^*)\Delta y_i = \int_a^b f(x(t), y(t))y'(t) \, \mathrm{d}t$$

Theorem 5.2.5.

$$\int_C P(x,y) \, \mathrm{d}x + \int_C Q(x,y) \, \mathrm{d}y = \int_C P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y$$

Example 5.2.2. Evaluate $\displaystyle\int_C y^2 \mathrm{d}x + x\mathrm{d}y$, where C is

1. A line segment from $(-5,-3)$ to $(0,2)$

   Answer.

   The equation of the line is $y + 3 = x + 5$.

   Set $y + 3 = x + 5 = t$. We get $y(t) = t - 3$ and $x(t) = t - 5$.

So, $\mathrm{d}y = \mathrm{d}t$ and $\mathrm{d}x = \mathrm{d}t$.

From $(-5, -3)$ to $(0, 2):\ 0 \leq t \leq 5$.

$$\begin{aligned}
\int_C y^2 \mathrm{d}x + x\mathrm{d}y &= \int_0^5 (t-3)^2 \mathrm{d}x + (t-5)\mathrm{d}y \\
&= \int_0^5 (t-3)^2 \mathrm{d}t + (t-5)\mathrm{d}t \\
&= \int_0^5 \left(t^2 + 9 - 6t + t - 5\right) \mathrm{d}t \\
&= \int_0^5 t^2 - 5t + 4 \,\mathrm{d}t \\
&= \left[\frac{1}{3}t^3 - \frac{5}{2}t^2 + 4t\right]_0^5 = -\frac{5}{6}
\end{aligned}$$

□

2. The parabola of $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$

Answer.

Let $y = t$, so $x(t) = 4 - t^2$.

So, $\mathrm{d}y = \mathrm{d}t$ and $\mathrm{d}x = -2t\mathrm{d}t$.

Since $-3 \leq y \leq 2$, we know $-3 \leq t \leq 2$. So,

$$\begin{aligned}
\int_C y^2 \mathrm{d}x + x\mathrm{d}y &= \int_{-3}^2 t^2(-2t)\mathrm{d}t + (4 - t^2)\mathrm{d}t \\
&= \int_{-3}^2 -2t^3 + 4t - t^2 \mathrm{d}t \\
&= \left[-\frac{1}{2}t^4 - \frac{1}{3}t^3 + 4t\right]_{-3}^2 = \frac{245}{6}.
\end{aligned}$$

□

Theorem 5.2.6. The line integral depends on the path in general. Line integral depends on the orientation of the path.

$$\int_{-C} f(x, y)\ \mathrm{d}s = -\int_C f(x, y)\ \mathrm{d}s.$$

Definition 5.2.3 (Vector Representation of Line Integrals). Let $\vec{\mathbf{r}}(t) = \langle x(t), y(t)\rangle = x(t)\hat{\mathbf{i}} +$

$y(t)\hat{\mathbf{j}}$. Then, $\vec{\mathbf{r}}'(t) = x'(t)\hat{\mathbf{i}} + y'(t)\hat{\mathbf{j}}$. So,

$$\int_C f(x,y)\,\mathrm{d}s = \int_a^b f(\vec{\mathbf{r}}(t))\left|\vec{\mathbf{r}}'(t)\right|\,\mathrm{d}t$$

Definition 5.2.4 (Line Integrals in Spaces).

$$\begin{aligned}\int_C f(x,y,z)\,\mathrm{d}s &= \int_a^b f(x,y,z)\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \\ &= \int_a^b f(\vec{\mathbf{r}}(t))\left|\vec{\mathbf{r}}'(t)\right|\,\mathrm{d}t,\end{aligned}$$

where $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$.

Theorem 5.2.7. Specially, if $f(x,y,z) = 1$, we have

$$L = \text{length of the curve } C = \int_C \mathrm{d}s = \int_a^b \left|\vec{\mathbf{r}}'(t)\right|\,\mathrm{d}t.$$

Example 5.2.3. Evaluate $\int_C y\sin z\,\mathrm{d}s$, where $C = \begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}$ $, 0 \le t \le 2\pi$ (the circular helix).

Answer.

$$x(t) = \cos t, \quad y(t) = \sin t, \quad z(t) = t, \quad 0 \le t \le 2\pi$$

$$x'(t) = -\sin t, \quad y'(t) = \cos t, \quad z'(t) = 1.$$

So,

$$\left|\vec{\mathbf{r}}'(t)\right| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{1+1} = \sqrt{2}.$$

$$\begin{aligned}\int_C y\sin z\,\mathrm{d}s \int_0^{2\pi} &\sin t\cdot\sin t(\sqrt{2})\,\mathrm{d}t \\ &= \sqrt{2}\int_0^2 \pi\sin^2 t\,\mathrm{d}t \\ &= \sqrt{2}\int_0^2 \pi\frac{1}{2}(1-\cos 2t)\,\mathrm{d}t \\ &= \frac{\sqrt{2}}{2}\left[t - \frac{1}{2}\sin 2t\right]_0^{2\pi} \\ &= \frac{2}{2}(2\pi) = \sqrt{2}\pi.\end{aligned}$$

□

Example 5.2.4. 1. Find the vector representation of the line segment starting at $(2,0,0)$ and ending at $(3,4,5)$.

Answer.

$$\begin{aligned}\vec{\mathbf{r}}(t) &= (1-t)\langle 2,0,0\rangle + t\langle 3,4,5\rangle, \qquad 0 \le t \le 1\\ &= \langle 2-2t+3t, 4t, 5t\rangle\\ &= \langle 2+t, 4t, 5, \rangle, \qquad 0 \le t \le 1.\end{aligned}$$

□

2. Evaluating $\int_C y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z$, where C is the line segment from the previous question.

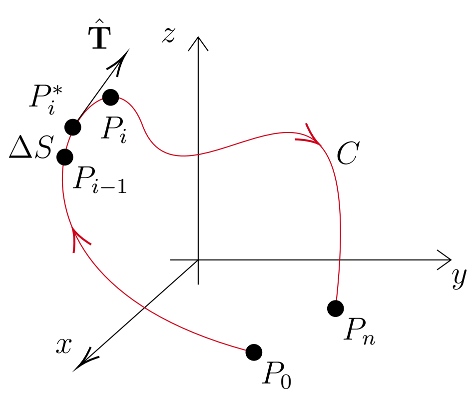
Answer.

$$x(t) = 2+t,\ \mathrm{d}x = \mathrm{d}t, \quad y(t) = 4t,\ \mathrm{d}y = 4\mathrm{d}t, \quad z(t) = 5t,\ \mathrm{d}z = 5\mathrm{d}t.$$

$$\begin{aligned}\int_C y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z &= \int_0^1 4t\mathrm{d}t + 5t(4)\mathrm{d}t + (2+t)(5)\mathrm{d}t\\ &= \int_0^1 29t + 10\,\mathrm{d}t\\ &= \left[\frac{29}{2}t^2 + 10t\right]_0^1\\ &= \frac{29}{2} + 10 = \frac{49}{2}.\end{aligned}$$

□

Definition 5.2.5 (Line Integrals of Vector Fields). Let $\vec{\mathbf{F}} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ be a continuous force field on $\mathbb{R}^3$. We want to compute the work done by this force in moving a particle along a smooth curve C.



So, we divide C into n sub-arc with length ΔS. Particles moves along curve C from P_{i-1} to P_i in the direction of the unit tangent vector $\hat{\mathbf{T}}(t_i^*)$ at P_i^*. The work done by the force $\vec{\mathbf{F}}$ in moving from P_{i-1} to P_i is

$$W \approx \vec{\mathbf{F}} \cdot \vec{\mathbf{D}} = \vec{\mathbf{F}}(x_i^*, y_i^*, z_i^*) \cdot \hat{\mathbf{T}}(t_i^*)\Delta S.$$

So,

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\vec{\mathbf{F}}(x_i^*, y_i^*, z_i^*) \cdot \hat{\mathbf{T}}(t_i^*)\right]\Delta S \\ &= \int_C \vec{\mathbf{F}}(x,y,z) \cdot \hat{\mathbf{T}}(x,y,z)\mathrm{d}s \\ &= \int_C \vec{\mathbf{F}} \cdot \hat{\mathbf{T}}\, \mathrm{d}s \end{aligned}$$

where $\hat{\mathbf{T}}$ is the unit tangent vector at the point (x, y, z).
Since $\mathrm{d}s = \left|\vec{\mathbf{r}}'(t)\right|\mathrm{d}t$ and $\hat{\mathbf{T}}(t) = \dfrac{\vec{\mathbf{r}}'(t)}{\left|\vec{\mathbf{r}}'(t)\right|}$, we have

$$\begin{aligned} W &= \int_a^b \left(\vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \frac{\vec{\mathbf{r}}'(t)}{\left|\vec{\mathbf{r}}'(t)\right|}\right) \cdot |\vec{\mathbf{r}}'(t)|\, \mathrm{d}t \\ &= \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, \mathrm{d}t \end{aligned}$$

Therefore, for a continuous vector field $\vec{\mathbf{F}}$ defined on a smooth curve C given by a vector function $\vec{\mathbf{r}}(t),\ a \le t \le b$, the line integral on $\vec{\mathbf{F}}$ along C is

$$\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, \mathrm{d}t = \int_C \vec{\mathbf{F}} \cdot \hat{\mathbf{T}}\, \mathrm{d}s.$$

Theorem 5.2.8. If $\vec{\mathbf{F}} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ is a vector field and $\vec{\mathbf{r}} = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$, then

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} &= \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, \mathrm{d}t \\ &= \int_a^b \langle P(x,y,z),\ Q(x,y,z),\ R(x,y,z)\rangle \cdot \langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\rangle\, \mathrm{d}t \\ &= \int_a^b \left(P(x,y,z)\frac{\mathrm{d}x}{\mathrm{d}t} + Q(x,y,z)\frac{\mathrm{d}y}{\mathrm{d}t} + R(x,y,z)\frac{\mathrm{d}z}{\mathrm{d}t}\right) \mathrm{d}t \\ &= \boxed{\int_a^b P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z} \end{aligned}$$

Example 5.2.5. Evaluate $\int_C \vec{\mathbf{F}} \mathrm{d}\vec{\mathbf{r}}$, where $\vec{\mathbf{F}} = xy\hat{\mathbf{i}} + yz\hat{\mathbf{j}} + zx\hat{\mathbf{k}}$ and $C = \begin{cases} x = t \\ y = t^2 \\ z = t^3 \end{cases}$, where $0 \leq t \leq 1$.

Answer.

$$x(t) = t,\ \mathrm{d}x = \mathrm{d}t; \quad y(t) = t^2,\ \mathrm{d}y = 2t\mathrm{d}t; \quad z(t) = t^3,\ dz = 3t^2\mathrm{d}t$$

So,

$$\begin{aligned} \int_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} &= \int_a^b P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z \\ &= \int_0^1 xy\mathrm{d}t + yz(2t)\mathrm{d}t + zx(3t^2)\mathrm{d}t \\ &= \int_0^1 t^3 + 5t^6 \,\mathrm{d}t \\ &= \left[\frac{1}{4}t^4 + \frac{5}{7}t^7\right]_0^1 = \frac{1}{4} + \frac{5}{7} = \frac{27}{28}. \end{aligned}$$

□

5.3 The Fundamental Theorem of Line Integral

Theorem 5.3.1 (The Fundamental Theorem of Line Integral).

$$\int_C \nabla f \cdot \mathrm{d}\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)),$$

where C is a smooth curve with vector function $\vec{\mathbf{r}}(t)$, with $a \leq t \leq b$ and f is a differentiable function of two or three variables whose gradient vector, ∇f, is continuous on C

Proof.

Let I be the line integral defined by

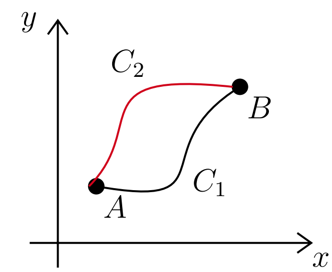
$$I = \int_C \nabla f \cdot \mathrm{d}\vec{\mathbf{r}}.$$

Then,

$$\begin{aligned} I &= \int_a^b \langle f_x(\vec{\mathbf{r}}(t)), f_y(\vec{\mathbf{r}}(t)), f_z(\vec{\mathbf{r}}(t))\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle \mathrm{d}t \\ &= \int_a^b (f_x(\vec{\mathbf{r}}(t))x'(t) + f_y(\vec{\mathbf{r}}(t))y'(t) + f_z(\vec{\mathbf{r}}(t))z'(t))\mathrm{d}t \\ &= \int_a^b \left(\frac{\partial f}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}\right)\mathrm{d}t \\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}(\vec{\mathbf{F}}(\vec{\mathbf{r}}(t))\mathrm{d}t = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)). \end{aligned}$$

■

Remark (Independence of Path). Let C_1 and C_2 be two paths that have the same initial and terminal points.



We know that, in general,

$$\int_{C_1} \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{r}} \neq \int_{C_2} \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{r}}$$

But we can show

$$\int_{C_1} \nabla f\cdot \mathrm{d}\vec{\mathbf{r}} = \int_{C_2} \nabla f\cdot \mathrm{d}\vec{\mathbf{r}}$$

The key difference here is that we may not be able to find a function f whose gradient $\nabla f = \vec{\mathbf{F}}$, the vector field.

Definition 5.3.1 (Conservative Vector Function). We say that vector function $\vec{\mathbf{F}}$ is conservative if there exists a function $f(x, y, z)$ such that $\nabla f = \vec{\mathbf{F}}$.

Theorem 5.3.2 (Testing Conservative). A vector field $\vec{\mathbf{F}} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ is conservative and P, Q, R have continuous first order partial derivatives if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

Theorem 5.3.3 (Independence of Path). The line integral of a conservative vector field depends only on initial and terminal points and is independent of path.

Definition 5.3.2 (Independence of Path). Let $\vec{\mathbf{F}}$ be a continuous vector field with domain D. We say that $\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}$ is independent of path if

$$\int_{C_1} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_{C_2} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}$$

for any two paths C_1 and C_2 in D that have the same initial and terminal points.

Lemma 5.1. Let $\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}$ be independent of path where C is a closed path, then $\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = 0$.

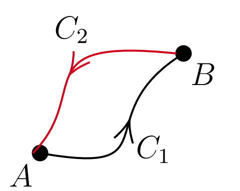
Proof.
Divide C into two paths, C_1 and C_2.
Then,

$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} &= \int_{C_1} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} \\ &= \int_{C_1} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} - \int_{-C_2} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}.\end{aligned}$$

Since $\vec{\mathbf{F}}$ is independent of path, we have

$$\int_{C_1} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_{-C_2} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}.$$

So, $\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = 0$.



■

Lemma 5.2. If $\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = 0$ for every closed path in D, then $\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}$ is independent of path in D.

Proof.
We have $\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = 0$ for any closed C in D.

$$\begin{aligned}0 = \int_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} &= \int_{C_1} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} \\ &= \int_{C_1} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} - \int_{-C_2} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}\end{aligned}$$

So, $\int_{C_1} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_{-C_2} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}$.
Therefore, $\vec{\mathbf{F}}$ is independent of path.

■

Theorem 5.3.4. From Lemma 5.1 and Lemma 5.2, we have $\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}$ is independent of path in D if and only if $\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = 0$ for every closed C in D.

Theorem 5.3.5 (Test for Conservation). If the vector field $\vec{\mathbf{F}} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ is conservative and P, Q, R have continuous first order partial derivatives, then the following is true:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}; \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}; \qquad \frac{\partial Q}{\partial Z} = \frac{\partial R}{\partial y}$$

Proof.
Since $\vec{\mathbf{F}}$ is conservative, there exists a function f such taht

$$\vec{\mathbf{F}} = \nabla f = f_x\hat{\mathbf{i}} + f_y\hat{\mathbf{j}} + f_z\hat{\mathbf{k}}.$$

So,

$$P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}} = f_x\hat{\mathbf{i}} + f_y\hat{\mathbf{j}} + f_z\hat{\mathbf{k}}.$$

That is,

$$\begin{cases} P = f_x \\ Q = f_y \\ R = f_z \end{cases} \implies \begin{cases} \dfrac{\partial P}{\partial y} = f_{yx} = f_{xy} = \dfrac{\partial f_y}{\partial x} = \dfrac{\partial Q}{\partial x} \\ \\ \dfrac{\partial P}{\partial Z} = f_{zx} = f_{xz} = \dfrac{\partial f_z}{\partial x} = \dfrac{\partial R}{\partial x} \\ \\ \dfrac{\partial Q}{\partial z} = f_{zy} = f_{yz} = \dfrac{\partial f_z}{\partial y} = \dfrac{\partial R}{\partial y} \end{cases}$$

■

Example 5.3.1. Consider the vector field

$$\vec{\mathbf{F}} = Ax\sin(\pi y)\hat{\mathbf{i}} + \left(x^2\cos(\pi y) + Bye^{-z}\right)\hat{\mathbf{j}} + y^2e^{-z}\hat{\mathbf{k}}.$$

1. For what values of A and B is the vector field $\vec{\mathbf{F}}$ conservative?

   Answer.

   We know: $P = Ax\sin(\pi y), \quad Q = (x^2\cos(\pi y) + Bye^{-z}), \quad R = y^2e^{-z}.$

Then, by Theorem 5.3.5, we should have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}; \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}; \qquad \frac{\partial Q}{\partial Z} = \frac{\partial R}{\partial y}.$$

From $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, we know $Ax\pi\sin(\pi y) = 2x\cos(\pi y)$, so $A = \frac{2}{\pi}$.
From $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$, we know $0 = 0$.
From $\frac{\partial Q}{\partial Z} = \frac{\partial R}{\partial y}$, we know $-Bye^{-z} = 2ye^{-z}$, and thus $B = -2$. Therefore,

$$\boxed{\vec{\mathbf{F}} = \frac{2x}{\pi}\sin(\pi y)\hat{\mathbf{i}} + \left(x^2\cos(\pi y) - 2ye^{-z}\right)\hat{\mathbf{j}} + y^2e^{-z}\hat{\mathbf{k}}}$$

Now, since $\vec{\mathbf{F}}$ is conservative, we can find an f such that $\nabla f = \vec{\mathbf{F}}$.
So, we have $\frac{\partial f}{\partial x} = \frac{2x}{\pi}\sin(\pi y)$.

$$f = \int \frac{2x}{\pi}\sin(\pi y)\,\mathrm{d}x + g(y,z) = \frac{x^2}{\pi}\sin(\pi y) + g(y,z).$$

Hence, $\frac{\partial f}{\partial y} = x^2\cos(\pi y) + \frac{\partial g}{\partial y} = x^2\cos(\pi y) - 2ye^{-z}$.

$$\begin{aligned}\frac{\partial g}{\partial y} &= -2ye^{-z}\\ g(y,z) &= \int -2ue^{-z}\,\mathrm{d}y + h(z)\\ g(y,z) &= -y^2e^{-z} + h(z).\end{aligned}$$

So,

$$f = \frac{x^2}{\pi}\sin(\pi y) - y^2e^{-z} + h(z)$$

So, $\frac{\partial f}{\partial z} = -\left(-y^2e^{-z}\right) + \frac{\mathrm{d}h}{\mathrm{d}z} = y^2e^{-z}$. Then, we would have $\frac{\mathrm{d}h}{\mathrm{d}z} = 0$, and thus $h(z) = 0$.
Therefore,

$$\boxed{f = \frac{x^2}{\pi}\sin(\pi y) - y^2e^{-z}}$$

□

2. Using your answer in the previous question to evaluate $\int_C \vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}}$, where C is

(a) The curve $\vec{\mathbf{r}} = \cos(t) + \hat{\mathbf{i}} + \sin(2t)\hat{\mathbf{j}} + \sin^2(t)\hat{\mathbf{k}}$.

Answer.

Since we have $\vec{\mathbf{r}}(0) = \langle 1,0,0\rangle = \hat{\mathbf{i}}$ and $\vec{\mathbf{r}}(2\pi) = \langle 1,0,0\rangle = \hat{\mathbf{i}}$, we know that $\vec{\mathbf{r}}(t)$ is a closed curve. Therefore, by Theorem 5.3.4, since $\vec{\mathbf{F}}$ is conservative, we have

$$\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = 0.$$

□

(b) Curve of intersection of the paraboloid $z = x^2 + 4y^2$ and the plane $z = 3x - 2y$ from $(0,0,0)$ to $\left(1, \frac{1}{2}, 2\right)$

Answer.

By Theorem 5.3.1, the Fundamental Theorem of Line Integral, we know

$$\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)).$$

So,

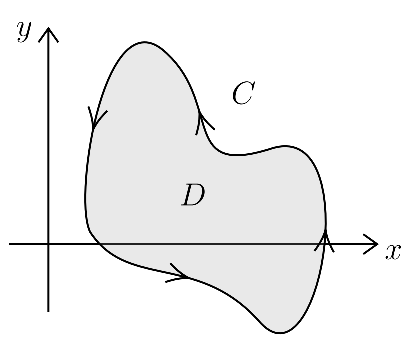
$$\begin{aligned}\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} &= \left[\frac{x^2}{\pi}\sin(\pi y) - y^2 e^{-z}\right]_{(0,0,0)}^{(1,1/2,2)} \\ &= \frac{1}{\pi} - \frac{1}{4e^2}.\end{aligned}$$

□

5.4 Green's Theorem

Green's theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane D bounded by C.

Definition 5.4.1 (Simply Connected Regions). Simply connected regions are regions that every simple closed curves in D enclosed only points that are in D.

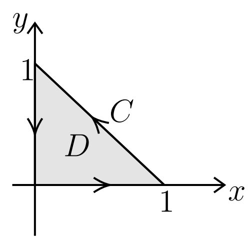


Theorem 5.4.1 (Green's Theorem). Let C be positively oriented piecewise-smooth simple closed curve in the plane, and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\oint_C P\mathrm{d}x + Q\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A.$$

Remark. "Positively oriented" means the direction is counter-clockwise.

Example 5.4.1. Evaluate $I = \oint_C x^4\mathrm{d}x + xy\mathrm{d}y$, where C is the following oriented triangle:



Answer.
By Green's Theorem, we have

$$I = \oint_C x^4\mathrm{d}x + xy\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A.$$

Since $P = x^4$ and $Q = xy$, we know $\frac{\partial Q}{\partial x} = y$ and $\frac{\partial P}{\partial y} = 0$. Therefore,

$$\begin{aligned} I = \iint_D (y-0)\,\mathrm{d}A = \int_0^1 \int_0^{1-x} y\mathrm{d}y\mathrm{d}x = \int_0^1 \left[\frac{1}{2}y^2\right]_0^{1-x} \mathrm{d}x &= \frac{1}{2}\left[\frac{1}{3}(1-x)^3\right]_0^1 \\ &= \frac{1}{6}\big((1-1)^3 - (0-1)^3\big) = \frac{1}{6}. \end{aligned}$$

□

Example 5.4.2. Evaluate $\oint_C \left(3y - e^{\sin x}\right)\mathrm{d}x + \left(7 + \sqrt{y^4+1}\right)\mathrm{d}y$ over C as $x^2 + y^2 = 9$.
Answer.

By Green's Theorem,

$$\begin{aligned}\oint_C \left(3y - e^{\sin x}\right)\mathrm{d}x + \left(7 + \sqrt{y^4+1}\right)\mathrm{d}y &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A \\ &= \iint_D (7-3)\,\mathrm{d}A \\ &= 4\iint_D \mathrm{d}A \\ &= 4A(D) = 4(9\pi) = 36\pi.\end{aligned}$$

□

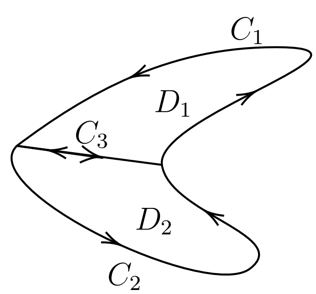
Remark (A Special Case). We can see that if $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1$, we have

$$\oint_C P\mathrm{d}x + Q\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}A = \iint_D \mathrm{d}A = A(D).$$

Also,

$$A(D) = \oint_C x\,\mathrm{d}y = -\oint_C y\,\mathrm{d}x = \frac{1}{2}\oint_C x\mathrm{d}y - y\mathrm{d}x.$$

Theorem 5.4.2 (Extension of Green's Theorem 1). We can extend Green's Theorem to finite union of simply connected regions:



$$\int_C P\mathrm{d}x + Q\mathrm{d}y = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A.$$

Proof.

Let $I = \displaystyle\int_C P\mathrm{d}x + Q\mathrm{d}y$. Then,

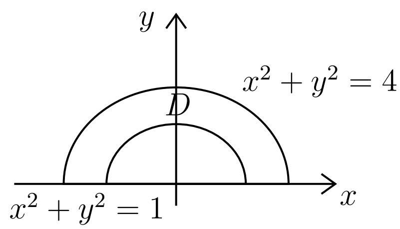
$$\begin{aligned} I &= \int_{C_1\cup C_3} P\mathrm{d}x + Q\mathrm{d}y + \int_{C_2\cup(-C_3)} P\mathrm{d}x + Q\mathrm{d}y \\ &= \iint_{D_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A + \iint_{D_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A \\ &= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A.\end{aligned}$$

■

Theorem 5.4.3 (Extension of Green's Theorem 2). Green's Theorem can be applied to regions with holes (regions that are not simply connected):

$$\int_C P\mathrm{d}x + Q\mathrm{d}y = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A.$$

Example 5.4.3. Evaluate $\oint_C y^2\mathrm{d}x + 3xy\mathrm{d}y$ along C as the following:



Answer.
Use the extension of the Green's Theorem:

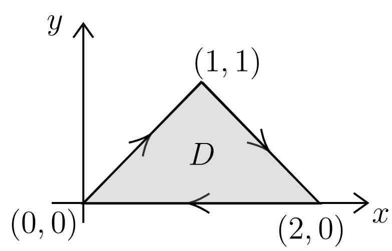
$$I = \oint_C y^2\mathrm{d}x + 3xy\mathrm{d}y = \iint_D (3y - 2y)\,\mathrm{d}A = \iint_D y\,\mathrm{d}A.$$

Change to polar coordinates: $1 \leq r \leq 2, \quad 0 \leq \theta \leq \pi, \quad y = r\sin\theta.$

$$\begin{aligned} I = \int_0^\pi \int_1^2 r\sin\theta \cdot r\,\mathrm{d}r\mathrm{d}\theta &= \int_0^\pi \sin\theta\,\mathrm{d}\theta \int_1^2 r^2\,\mathrm{d}r \\ &= \Big[-\cos\theta\Big]_0^\pi \left[\frac{1}{3}r^3\right]_1^2 \\ &= (-(-1) - (-1))\left(\frac{8}{3} - \frac{1}{3}\right) \\ &= 2\left(\frac{7}{3}\right) = \frac{14}{3}. \end{aligned}$$

□

Example 5.4.4. Evaluate $\oint_C (x^2 - xy)\mathrm{d}x + (xy - x^2)\mathrm{d}y$, where C is given by the following triangle.



Answer.

This question is left as an exercise so the steps are omitted, but the answer should be

$$I = -\frac{4}{3}.$$

□

5.5 Curl and Divergence

Definition 5.5.1 (Divergence and Curl). For a vector field $\vec{\mathbf{F}} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$, we define divergence and curl as

$$\begin{aligned} \operatorname{div} \vec{\mathbf{F}} = \nabla \cdot \vec{\mathbf{F}} &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle \\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \end{aligned}$$

$$\begin{aligned} \operatorname{curl} \vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial Z} \right) \hat{\mathbf{i}} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{\mathbf{k}} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial Z} \right) \hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{\mathbf{k}} \end{aligned}$$

Example 5.5.1. Find the divergence and curl of the vector field

$$\vec{\mathbf{F}} = xy\hat{\mathbf{i}} + \left(y^2 - z^2\right)\hat{\mathbf{j}} + yz\hat{\mathbf{k}}$$

Answer.

$$\begin{aligned} \operatorname{div} \vec{\mathbf{F}} = \nabla \cdot \vec{\mathbf{F}} &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle xy, \left(y^2 - z^2\right), yz \right\rangle \\ &= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}\left(y^2 - z^2\right) + \frac{\partial}{\partial z}(yz) \\ &= y + 2y + y = 4y. \end{aligned}$$

$$\begin{aligned}
\text{curl } \vec{\mathbf{F}} = \boldsymbol{\nabla} \times \vec{\mathbf{F}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & y^2 - z^2 & yz \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}\left(y^2 - z^2\right) \right)\hat{\mathbf{i}} + (0-0)\hat{\mathbf{j}} + \left(\frac{\partial}{\partial x}\left(y^2 - z^2\right) - \frac{\partial}{\partial y}(xy) \right)\hat{\mathbf{k}} \\
&= (z + 2z)\hat{\mathbf{i}} - 0 + (0 - x)\hat{\mathbf{k}} \\
&= 3z\hat{\mathbf{i}} - x\hat{\mathbf{k}}.
\end{aligned}$$

□

Theorem 5.5.1 (Properties of Curl, Divergence, and Gradient). Let f be a scalar field and $\vec{\mathbf{F}}$ be a vector field. Suppose f and $\vec{\mathbf{F}}$ are all smooth and have all partial derivatives continuous, then

1. $\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \times \vec{\mathbf{F}} \right) = 0$ or in words, div $(\text{curl } \vec{\mathbf{F}}) = 0$

   Proof.

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \times \vec{\mathbf{F}} \right) &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle \\
&= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \\
&= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} \\
&= 0
\end{aligned}$$

■

2. $\boldsymbol{\nabla} \times (\nabla f) = 0$ or in words, $\boldsymbol{\nabla} \times (\text{gradient } f) = 0$

   Proof.

$$\boldsymbol{\nabla} \times (\nabla f) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix}$$

$$\begin{aligned}
&= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right)\hat{\mathbf{i}} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \right)\hat{\mathbf{j}} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right)\hat{\mathbf{k}} \\
&= 0
\end{aligned}$$

■

Remark. If $\vec{\mathbf{F}}$ is conservative, then $\vec{\mathbf{F}} = \nabla f$ and

$$\text{curl}\,\vec{\mathbf{F}} = \text{curl}\,(\nabla f) = 0.$$

Theorem 5.5.2. If $\vec{\mathbf{F}}$ is a vector field on $\mathbb{R}^3$ and its component functions, P, Q, and R, have continuous partial derivatives and curl $\vec{\mathbf{F}} = 0$, then $\vec{\mathbf{F}}$ is conservative.

Example 5.5.2. Show that

$$\vec{\mathbf{F}}(x, y, z) = y^2 z^3 \hat{\mathbf{i}} + 2xyz^3 \hat{\mathbf{j}} + 3xy^2 z^2 \hat{\mathbf{k}}$$

is a conservative field and find a function f such that $\vec{\mathbf{F}} = \nabla f$.

Answer.

Note that

$$\text{curl}\,\vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 z^3 & 2xyz^3 & 3xy^2 z^2 \end{vmatrix} = 0$$

Also, y^2z^3, $2xyz^3$, and $3xy^2z^2$ are in $\mathbb{R}^3$ and have continuous partial derivatives.
Therefore, by Theorem 5.5.2, $\vec{\mathbf{F}}$ is conservative.
Now, we can find the f such that $\nabla f = \vec{\mathbf{F}}$.
So,

$$\frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} = y^2 z^3 \hat{\mathbf{i}} + 2xyz^3 \hat{\mathbf{j}} + 3xy^2 z^2 \hat{\mathbf{k}}$$

That is,

$$\frac{\partial f}{\partial x} = y^2 z^3; \qquad \frac{\partial f}{\partial y} = 2xyz^3; \qquad \frac{\partial f}{\partial z} = 3xy^2 z^2.$$

From $\dfrac{\partial f}{\partial x} = y^2 z^3$, we have $f = xy^2 z^3 + g(y, z)$
So,

$$\frac{\partial f}{\partial y} = 2xyz^3 + \frac{\partial g}{\partial y} = 2xyz^3.$$

We have $\dfrac{\partial g}{\partial y} = 0$, which means $g(y, z) = h(z)$.
So,

$$\frac{\partial f}{\partial z} = 3xy^2 z^2 + \frac{\mathrm{d}h}{\mathrm{d}z} = 3xy^2 z^2$$

Similarly, $\dfrac{\mathrm{d}h}{\mathrm{d}z} = 0$, so $h(z)$ is a constant function.
Hence,

$$\boxed{f = xy^2 z^3 + C}$$

□

Definition 5.5.2 (Laplace Operator/Laplacian). The Laplace operator (or laplacian) is denoted as $\nabla \cdot \nabla$ or ∇^2 and is defined by

$$\nabla^2 = \left\langle \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2} \right\rangle$$

Theorem 5.5.3 (More Properties). Let f and g be scalar fields and $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ be vector fields. Define

$$(f\vec{\mathbf{F}})(x, y, z) = f(x, y, z)\vec{\mathbf{F}}(x, y, z)$$
$$(\vec{\mathbf{F}} \cdot \vec{\mathbf{G}})(x, y, z) = \vec{\mathbf{F}}(x, y, z) \cdot \vec{\mathbf{G}}(x, y, z)$$
$$(\vec{\mathbf{F}} \times \vec{\mathbf{G}}) = \vec{\mathbf{F}}(x, y, z) \times \vec{\mathbf{G}}(x, y, z)$$

Suppose f, g, $\vec{\mathbf{F}}$ and $\vec{\mathbf{G}}$ are all smooth and have all partial derivatives continuous, then

1. $\boldsymbol{\nabla} \cdot \left(\vec{\mathbf{F}} + \vec{\mathbf{G}}\right) = \boldsymbol{\nabla} \cdot \vec{\mathbf{F}} + \boldsymbol{\nabla} \cdot \vec{\mathbf{G}}$
2. $\boldsymbol{\nabla} \times \left(\vec{\mathbf{F}} + \vec{\mathbf{G}}\right) = \boldsymbol{\nabla} \times \vec{\mathbf{F}} + \boldsymbol{\nabla} \times \vec{\mathbf{G}}$
3. $\boldsymbol{\nabla} \cdot \left(f\vec{\mathbf{F}}\right) = f\boldsymbol{\nabla} \cdot \vec{\mathbf{F}} + \vec{\mathbf{F}} \cdot \nabla f$
4. $\boldsymbol{\nabla} \times \left(f\vec{\mathbf{F}}\right) = f\boldsymbol{\nabla} \times \vec{\mathbf{F}} + (\nabla f) \times \vec{\mathbf{F}}$
5. $\boldsymbol{\nabla} \cdot \left(\vec{\mathbf{F}} \times \vec{\mathbf{G}}\right) = \vec{\mathbf{G}} \cdot \boldsymbol{\nabla} \times \vec{\mathbf{F}} - \vec{\mathbf{F}} \cdot \boldsymbol{\nabla} \times \vec{\mathbf{G}}$
6. $\boldsymbol{\nabla} \cdot (\nabla f \times \nabla g) = 0$
7. $\boxed{\boldsymbol{\nabla} \times \left(\boldsymbol{\nabla} \times \vec{\mathbf{F}}\right) = \nabla(\boldsymbol{\nabla} \cdot \vec{\mathbf{F}}) - \nabla^2\vec{\mathbf{F}}}$

Theorem 5.5.4 (Stoke's Theorem). Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\vec{\mathbf{F}}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S, then

$$\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \iint_S \boldsymbol{\nabla} \times \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}}$$